Title: Linear equations with constant coefficients

Name of teacher: R. M. Wayal
Assistant professor
Hutatma Rajguru Mahavidyalaya, Rajgurunagar
This study material is useful to the students of
Subject: Mathematics
Course/Class: B.Sc. / T. Y. B. Sc.
Semester:III
Paper No.: MT-335, Ordinary Differential Equations

Topic: Solution of homogeneous linear differential equations

Linear Equations:
A linear differential equation of \( n \)th order is
\[
a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x) \frac{dy}{dx} + a_n(x)y = R(x) \quad \ldots (1)
\]
Where \( a_0(x) \neq 0 \)

Case1: If the coefficients \( a_o(x), a_1(x), \ldots, a_n(x) \) are functions of \( x \) then equation (1) is called as linear differential equation with variable coefficients.

Case2: If the coefficients \( a_o(x), a_1(x), \ldots, a_n(x) \) are constant then equation (1) is called as linear differential equation with constant coefficients.

Any linear differential equation with constant coefficients is given by
\[
a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = R(x)
\]
Where \( a_0 \neq 0, a_1, a_2, \ldots, a_n \) are complex constants, and \( R(x) \) is some complex valued function on an interval \( I \).
By dividing \( a_0 \) we get an equation of same form with \( a_0 = 1 \). Therefore we can always assume \( a_0 = 1 \)
i.e.

\[
\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = R(x) \ldots (2)
\]

If \( R(x) = 0 \) then equation (2) is called homogeneous linear differential equation.

If \( R(x) \neq 0 \) then equation (2) is called non-homogeneous linear differential equation.

**The operator** \( D \):

\( D \) is the differential operator which differentiates the function on which it operates
i.e. \( D y = \frac{dy}{dx} \), \( D^2 y = \frac{d^2 y}{dx^2}, \ldots, D^n y = \frac{d^n y}{dx^n} \)

Therefore the equation

\[
\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0
\]

Can be written as

\[
D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} D y + a_n y = 0
\]
i.e.

\[
D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n \) \) y = 0
\]

\( f(D) y = 0 \)

Where \( f(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n \)

**Auxiliary equation or characteristic equation:**

If \( f(D) y = 0 \) is homogeneous linear differential equation with constant coefficients
i.e.

\[
D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} D y + a_n y = 0
\]

Then corresponding auxiliary equation is
\[ f(m) = 0 \]
i.e.
\[ m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \]

**Roots of auxiliary equation:**

1) **Real and distinct:**

   If the auxiliary equation \( f(m) = 0 \) of \( n^{th} \) order linear differential equation \( f(D)y = 0 \) has \( n \) distinct roots say \( m_1, m_2, \ldots, m_n \) then its \( n \) linearly independent solutions are:

   \[ e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x} \]

   Hence general solution of \( f(D)y = 0 \) is

   \[ y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \]

Example 1: Find the general solution of \( (D^2 - D - 6)y = 0 \).
Solution: Auxiliary equation is

\[ m^2 - m - 6 = 0 \]
\[ \therefore (m - 3)(m - 2) = 0 \]
\[ \therefore m - 3 = 0 \text{ and } m - 2 = 0 \]
\[ \therefore m = 3 \text{ and } m = 2 \]

Roots are real and distinct
Hence general solution is

\[ y = c_1 e^{3x} + c_2 e^{2x} \]

Example 2: Find the general solution of \( (D^3 + 2D^2 - 15D)y = 0 \).
Solution: Auxiliary equation is

\[ m^3 + 2m^2 - 15m = 0 \]
\[ \therefore m(m^2 + 2m - 15) = 0 \]
\[ \therefore m(m + 5)(m - 3) = 0 \]
\[ m = 0, -5, 3 \]

Real and distinct roots
Hence general solution is
\[ y = c_1 e^{0x} + c_2 e^{-5x} + c_3 e^{3x} \]
\[ y = c_1 + c_2 e^{-5x} + c_3 e^{3x} \]

Example 3: Find the general solution of \((D^3 - D^2 - 4D + 4)y = 0\).

Solution: Auxiliary equation is
\[ m^3 - m^2 - 4m + 4 = 0 \]

\( m = 1 \) satisfy the above equation therefore \( m = 1 \) is one root of this equation. Find other roots by using synthetic division.

\[
\begin{array}{cccc|c}
1 & 1 & -1 & -4 & 4 \\
 & 1 & 0 & -4 \\
1 & 0 & -4 & 0 \\
\end{array}
\]

\[ m^3 - m^2 - 4m + 4 = (m - 1)(m^2 - 4) = 0 \]
\[ \therefore (m - 1)(m - 2)(m + 2) = 0 \]
\[ m = 1, 2, -2 \]

General solution is
\[ y = c_1 e^x + c_2 e^{2x} + c_3 e^{-2x} \]

2) Real and repeated roots:

Suppose root of an auxiliary equation \( f(m) = 0 \) is repeated two times, say \( m = a, a \) are roots of auxiliary equation then two linearly independent solutions are \( e^{ax}, xe^{ax} \) and the corresponding term in general solution of \( f(D)y = 0 \) is
\[ c_1 e^{ax} + c_2 xe^{ax} \]
If \( m = a, a, a \) are repeated three times then three linearly independent solutions are \( e^{ax}, xe^{ax}, x^2 e^{ax} \) and the corresponding term in general solution of \( f(D)y = 0 \) is 
\[
c_1 e^{ax} + c_2 xe^{ax} + c_3 x^2 e^{ax}
\]

If \( m = a, a, a, a \) are repeated four times and \( m = b, b, b, b \) are repeated five times then nine linearly independent solutions are \( e^{ax}, xe^{ax}, x^2 e^{ax}, x^3 e^{ax}, e^{bx}, xe^{bx}, x^2 e^{bx}, x^3 e^{bx}, x^4 e^{bx} \) and the corresponding term in general solution of \( f(D)y = 0 \) is 
\[
c_1 e^{ax} + c_2 xe^{ax} + c_3 x^2 e^{ax} + c_4 x^3 e^{ax} + c_5 e^{bx} + c_6 xe^{bx} + c_7 x^2 e^{bx} + c_8 x^3 e^{bx} + c_9 x^4 e^{bx}
\]

Example 1: Find the general solution of \( (D^3 - 4D^2 + 4D)y = 0 \)
Solution: Auxiliary solution is 
\[
m^3 - 4m^2 + 4m = 0
\]
\[
\therefore m(m^2 - 4m + 4) = 0
\]
\[
\therefore m(m - 2)(m - 2) = 0
\]
\[
m = 0, 2, 2 \text{ are roots of auxiliary equation}
\]
The general solution is 
\[
y = c_1 e^{0x} + c_2 e^{2x} + c_3 xe^{2x}
\]
\[
y = c_1 + c_2 e^{2x} + c_3 xe^{2x}.
\]

Example 2: Find the general solution of \( (2D^4 - 5D^3 - 3D^2)y = 0 \).
Solution: Auxiliary solution is 
\[
2m^4 - 5m^3 - 3m^2 = 0
\]
\[
\therefore m^2(2m^2 - 5m - 3) = 0
\]
\[
\therefore m^2(2m^2 - 6m + 1m - 3) = 0
\]
\[
\therefore m^2[2m(m - 3) + 1(m - 3)] = 0
\]
\[
\therefore m^2(2m + 1)(m - 3) = 0
\]
\[
\therefore m = 0, 0, -\frac{1}{2}, 3
\]
Are roots of auxiliary equation. Therefore general solution is
\[ y = c_1 e^{0x} + c_2 xe^{0x} + c_3 e^{\frac{1}{2}x} + c_4 e^{3x} \]

\[ y = c_1 + c_2 x + c_3 e^{\frac{1}{2}x} + c_4 e^{3x}. \]

Example 2: Find the general solution of
\[ (D^4 + 3D^3 - 6D^2 - 28D - 24)y = 0. \]

Solution: Auxiliary solution is
\[ m^4 + 3m^3 - 6m^2 - 28m - 24 = 0 \]
\[ m = -2 \Rightarrow 16 - 24 - 24 + 56 - 24 = 72 - 72 = 0 \]
Therefore \( m = -2 \) is one root

\[
\begin{array}{ccc}
-2 & 1 & 3 \\
-2 & -2 & 16 \\
1 & 1 & -8
\end{array}
\]

\[ m^4 + 3m^3 - 6m^2 - 28m - 24 = (m + 2)(m^3 + m^2 - 8m - 12) = 0 \ldots (1) \]

Now find roots of \( m^3 + m^2 - 8m - 12 = 0 \)

\[ m = -2 \Rightarrow -8 + 4 + 16 - 12 = 0 \]
\[ m = -2 \] is root of above equation
\[ m^3 + m^2 - 8m - 12 = (m + 2)(m^2 - m - 6) = 0 \]

From equation (1)
\[ m^4 + 3m^3 - 6m^2 - 28m - 24 = (m + 2)(m + 2)(m^2 - m - 6) = 0 \]
\[ (m + 2)(m + 2)(m + 2)(m - 3) = 0 \]
\[ m = -2, -2, -2, 3 \]
are roots of auxiliary equation

The general solution is
\[ y = c_1 e^{-2x} + c_2 xe^{-2x} + c_3 x^2 e^{-2x} + c_4 e^{3x}. \]

3) Complex roots:
Consider a linear differential equation \( f(D)y = 0 \)
Let \( f(m) = 0 \) be its auxiliary equation. If \( a \pm ib \) are roots of auxiliary equation then general solution of \( f(D)y = 0 \) is
\[ y = e^{ax} (c_1 \cos bx + c_2 \sin bx) \]

If \( a \pm ib \) are repeated two times then general solution is
\[ y = e^{ax} [(c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx] \]

Example 1: Find the general solution of \( (D^2 - 2D + 2)y = 0 \).
Solution: Auxiliary equation is
\[ m^2 - 2m + 2 = 0 \]
\[ \therefore m = \frac{2 \pm \sqrt{4 - 8}}{2} \]
\[ \therefore m = \frac{2 \pm \sqrt{-4}}{2} \]
\[ \therefore m = \frac{2 \pm 2i}{2} \]
\[ \therefore m = 1 \pm i \]

\( m = 1 \pm i \) are roots of auxiliary equation

Therefore general solution is
\[ y = e^x (c_1 \cos x + c_2 \sin x). \]
Example 2: Find the general solution of \((D^4 + 2D^3 + 10D^2)y = 0\).

Solution: auxiliary equation is

\[ m^4 + 2m^3 + 10m^2 = 0 \]
\[ \therefore m^2(m^2 + 2m + 10) = 0 \]
\[ \Rightarrow m^2 = 0 \text{ and } m^2 + 2m + 10 \]
\[ m = 0,0 \text{ and } m = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i \]

\( m = 0,0,-2 \pm 3i \) are roots of auxiliary equation

Therefore general solution is

\[ y = c_1 e^{0x} + c_2 xe^{0x} + e^{-2x}(\cos 3x + \sin 3x) \]
\[ y = c_1 + c_2 x + e^{-2x}(\cos 3x + \sin 3x). \]

Example 3: Solve the differential equation \((D^5 + 2D^3 + D)y = 0\).

Solution: Auxiliary equation is

\[ m^5 + 2m^3 + m = 0 \]
\[ \therefore m(m^4 + 2m^2 + 1) = 0 \]
\[ \therefore m (m^2 + 1)^2 = 0 \]
\[ m = 0, m^2 + 1 = 0, m^2 + 1 = 0 \]
\[ m = 0, m = \pm i, m = \pm i \]

\( m = 0, \pm i, \pm i \) are roots of equation

Therefore general solution is

\[ y = c_1 e^{0x} + e^{0x}[(c_2 + c_3 x) \cos x + (c_4 + c_5 x) \sin x] \]
\[ y = c_1 + (c_2 + c_3 x) \cos x + (c_4 + c_5 x) \sin x. \]

**Exercise**

Find the general solution of following differential equations

1) \((D^3 - 3D^2 - D + 3)y = 0\)
2) \((10D^3 + D^2 - 7D + 2)y = 0\)
3) \((D^2 + 5D + 6)y = 0\)
4) \((9D^3 + 6D^2 + D)y = 0\)
5) \((4D^3 - 27D + 27)y = 0\)
6) \((D^3 + 5D^2 + 3D - 9)y = 0\)
7) \((D^2 - 4D + 7)y = 0\)
8) \((D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0\)
9) \((D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0\)
10) \((D^4 + 5D^2 + 4)y = 0\).

References:

1) Elementary Differential Equations, Rainville and Bedient, Macmillan Publication.