Commutative Algebra

Vineet Joshi
Dept. of Mathematics
S.P. Pune University
Hilbert Basis Theorem
Theorem -: If $A$ is Noetherian, so is $A[x]$.  

Proof -: Assume that $A$ is Noetherian.  
Let $I$ be an ideal of $A[x]$. To prove $A[x]$ is Noetherian we have to show that $I$ is finitely generated.  
Assume on the contrary that $I$ is not f.g.  

Then $I \neq (0)$. Let $f_1$ be a polynomial in $I$ which is of the least degree, say $d_1$, in $I$. Note that $(f_1) \subsetneq I$, otherwise $I$ is f.g.  
Now, choose a polynomial $f_2 \in I \setminus (f_1)$ such that $\deg(f_2) = d_2$ is least in $I \setminus (f_1)$. 

Clearly, $d_1 \leq d_2$. Further, note that $(f_1, f_2) \not\subset I$, otherwise $I$ is f.g. Now choose another polynomial $f_3 \in I \setminus (f_1, f_2)$ s.t. $\deg(f_3) = d_3$ is least in $I \setminus (f_1, f_2)$. Clearly $d_1 \leq d_2 \leq d_3$.

Continuing in this fashion, we get a seq of polynomials $f_1, f_2, \ldots, f_n, \ldots$ s.t.

1. $d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_n \leq \ldots$
2. $f_n$ is a polynomial of least degree in $I \setminus (f_1, f_2, \ldots, f_{n-1})$.

Let $f_i(x) = a_i x^{d_i} + \text{lower deg. terms}$, where $a_i \in A$

$i = 1, 2, \ldots, n, \ldots$
It is clear that \((a_1) \subseteq (q_1, q_2) \subseteq (q_1, q_2, q_3) \subseteq \ldots\).

Since \(A\) is Noetherian, we get
\[
(q_1, q_2, \ldots, q_n) = (q_1, q_2, \ldots, q_n, q_{n+1}) = (q_1, q_2, \ldots, q_{n+1}, q_{n+2}) = \ldots
\]

Since \(q_{n+1} \in (q_1, \ldots, q_{n+1}) = (q_1, \ldots, q_n)\), we have

\[
q_{n+1} = b_1 q_1 + b_2 q_2 + \ldots + b_n q_n \quad \text{①}
\]

Where \(b_i \in A, i = 1, 2, \ldots, n\). Consider the polynomial
\[
g(x) = f_{n+1}(x) - \left( b_1 f_1(x) \cdot x^{(d_{n+1} - d_1)} + b_2 f_2(x) \cdot x^{(d_{n+1} - d_2)} + \ldots + b_n f_n(x) \cdot x^{(d_{n+1} - d_n)} \right) \quad ②
\]
\[ g(x) = f_{n+1}(x) - \left( b_1 f_1(x) \cdot x + b_2 f_2(x) \cdot x \left( d_{n+1} - d_n \right) \right) - \cdots - b_n f_n(x) \cdot x \left( d_{n+1} - d_n \right) \]

\[ = \left( a_{n+1} x^{d_{n+1}} + \text{lower deg. terms} \right) \]

\[ - \left( b_1 \left( a_1 x^{d_1} + \text{lower deg. terms} \right) \cdot x \left( d_{n+1} - d_1 \right) \right) - \cdots - b_n \left( a_n x^{d_n} + \text{lower deg. terms} \right) \cdot x \left( d_{n+1} - d_n \right) \]

\[ = \left( a_{n+1} - a_1 b_1 - a_2 b_2 - \cdots - a_n b_n \right) x^{d_{n+1}} + \text{lower deg. terms} \]

\[ = 0 + \text{lower deg. terms starting from } x^{d_n} \text{ (as first term is zero by 0)} \]
Hence $g(x)$ is a polynomial of degree $< d_{n+1}$.

We observe that $g(x) \notin \mathbb{I}(f_1, f_2, \ldots, f_n)$. Otherwise, this will contradict the choice of the poly.

$$f_{n+1}(x) \in \mathbb{I}(f_1, f_2, \ldots, f_n)$$

such that

$$\deg(f_{n+1}(x)) = d_{n+1}$$

is least in $\mathbb{I}(f_1, \ldots, f_n)$.

Hence $g(x) \in (f_1, f_2, \ldots, f_n)$.

From (2), it is clear that $f_{n+1} \in (f_1, f_2, \ldots, f_n)$.

Again a contradiction to (3). Hence $I$ is fg. This proves that $A[x]$ is Noetherian.
Note that if $A[x]$ is Noetherian then $A$ is Noetherian. (Prove this). Hence we have

Cor. 1 \ $A$ is Noetherian $\iff$ $A[x]$ is Noetherian.

Cor. 2 \ $A$ is Noetherian $\iff$ $A[x_1, x_2, \ldots, x_n]$ Noetherian.

Q: Is an analogue of Hilbert Basis Thm true for Artinian rings?

A: No

Suppose $R$ is not the zero ring. Let $\mathfrak{p}$ be a prime ideal of $R$. Then $\mathfrak{p}[x]$, the ideal of polynomials with coefficients in $\mathfrak{p}$, is a prime ideal of $R[x]$. But $(\mathfrak{p}[x], x)$ is a prime ideal properly containing $\mathfrak{p}[x]$ so we see that $\dim(R[x]) > \dim(R) \geq 0$. So dim $R[x]$ is always at least 1 and $R[x]$ is never ever Artinian (unless $R$ is the zero ring).
Example  We shall show that $2\mathbb{Z}$ is Noetherian but $2\mathbb{Z}[x]$ is not.

Let $I$ be a non-zero ideal of $2\mathbb{Z}$. If $a \in I$ then $-a \in I$ because $I$ is a subgroup of $(2\mathbb{Z}, +)$. Let $a$ be the smallest positive element of $I$. Suppose that $b \in I$ with $b > 0$. In $\mathbb{Z}$, there are integers $q$ and $r$ such that $b = qa + r$ and $0 \leq r < a$. Now,

$$qa = a + a + \ldots + a,$$

$q$ times

which is in $I$. Thus $b - qa \in I$ and so $r \in I$. Because $0 \leq r < a$, we must have $r = 0$. Therefore $I = \{qa : q \in \mathbb{Z}\}$ (of course, $a$ is even).

So every ideal of $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$. We know that $\mathbb{Z}$ satisfies ACC, so $2\mathbb{Z}$ must also satisfy ACC.

If $2\mathbb{Z}[x]$ is Noetherian then it is finitely generated. Suppose that the generators are $f_1(x), \ldots, f_n(x)$, where $f_i(x)$ has degree $d_i$. Put $N = \max \{d_1, \ldots, d_n\}$. Then if $g(x) \in \langle \{f_1(x), \ldots, f_n(x)\} \rangle$ then

either  the degree of $g(x)$ is at most $N$

or  every coefficient in $g(x)$ is divisible by 4.

Therefore $2x^{N+1} \notin \langle \{f_1(x), \ldots, f_n(x)\} \rangle$ but $2x^{N+1} \in 2\mathbb{Z}[x]$. This contradiction shows that $2\mathbb{Z}[x]$ is not Noetherian.
Corollary: If $M_1, \ldots, M_n$ are Noetherian [Artinian] $A$-modules then so is

$$M_1 \oplus \ldots \oplus M_n.$$

Proof: This follows by induction and the previous Theorem applied to the exact sequence

$$0 \rightarrow M_n \rightarrow \bigoplus_{i=1}^{n} M_i \rightarrow \bigoplus_{i=1}^{n-1} M_i \rightarrow 0$$

where

$$\alpha : x \mapsto (0, \ldots, 0, x)$$

$$\beta : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}).$$
Corollary: Any homomorphic image of a Noetherian [Artinian] ring is Noetherian [Artinian].

Ring and module properties can be linked:

Theorem: Let $A$ be a Noetherian [Artinian] ring and $M$ a finitely generated $A$-module. Then $M$ is Noetherian [Artinian].
We know that

\[ M \cong A^n / N \]

for some \( n > 0 \) and some submodule \( N \) of \( A^n \).

But \( A^n \) is Noetherian [Artinian], being a direct sum of Noetherian [Artinian] modules.

Hence, by the previous Corollary, \( M \) is Noetherian [Artinian].