Commutative Algebra

Vinayak Joshi
Dept. of Mathematics
S P Pune University
EXTENDED AND CONTRACTED IDEALS IN RINGS OF FRACTIONS

Let \( A \) be a ring, \( S \) a multiplicatively closed subset of \( A \) and \( f: A \to S^{-1}A \) the natural homomorphism, defined by \( f(a) = a/1 \). Let \( C \) be the set of contracted ideals in \( A \), and let \( E \) be the set of extended ideals in \( S^{-1}A \) (cf. (1.17)). If \( a \) is an ideal in \( A \), its extension \( a^e \) in \( S^{-1}A \) is \( S^{-1}a \) (for any \( y \in a^e \) is of the form \( \sum a_i/s_i \), where \( a_i \in a \) and \( s_i \in S \); bring this fraction to a common denominator).

**Proposition 3.11.** i) Every ideal in \( S^{-1}A \) is an extended ideal.

ii) If \( a \) is an ideal in \( A \), then \( a^{ec} = \bigcup_{s \in S} (a:s) \). Hence \( a^e = (1) \) if and only if \( a \) meets \( S \).

iii) \( a \in C \iff \) no element of \( S \) is a zero-divisor in \( A/a \).

iv) The prime ideals of \( S^{-1}A \) are in one-to-one correspondence \( (\wp \leftrightarrow S^{-1}\wp) \) with the prime ideals of \( A \) which don’t meet \( S \).
Proof:

Let \( J \) be an ideal of \( \mathbb{Z}A \).

Let \( I = \{ a \in A \mid \frac{a}{1} \in J \} \). We claim that \( I \) is an ideal of \( A \).

Since \( \frac{0}{1} \in J \), we have \( 0 \in I \). Thus \( I \) is nonempty. Let \( a, b \in I \). Then \( a \cdot b \in J \).

Since \( \frac{1}{1} \in J \) is an ideal, we have

\[
\frac{a + b}{1} = \frac{a}{1} + \frac{b}{1} \in J \Rightarrow a + b \in I.
\]

Let \( a \in I \) and \( x \in A \). Then \( \frac{x}{1} \in J \).

Being \( J \) an ideal of \( \mathbb{Z}A \), \( \frac{1}{1} \cdot a = \frac{xa}{1} \in J \).
Hence $ra \in I$. This proves that $I$ is an ideal of $A$.

Now, we claim that $J = I^e \subseteq I$.

Let $\frac{a}{b} \in J$. Since $\frac{b}{1} \in S^{-1}A$, we have

\[
\frac{b}{1} \cdot \frac{a}{b} = \frac{a}{1} \in J. \text{ This gives } a \in I.
\]

Hence $f(a) = \frac{a}{1} \in f(I)$. Take $\frac{1}{b} \in S^{-1}A$.

\[
\Rightarrow \frac{a}{b} \cdot \frac{1}{b} \in I^e \Rightarrow \frac{a}{b} \in I^e. \text{ This gives } \boxed{J \subseteq I^e}.
\]
Conversely, assume that $a/b \in I^e$. Then

$$\frac{a}{b} = \sum_{\text{finite}} \frac{\xi_i}{d_i} \cdot f(x_i), \quad \xi_i \in S intending b\text{ to } x_i \in I. $$

Since $a_i \in I$, $f(x_i) = x_i \in J$ and $J$ is an ideal of $S$, $A$ gives $\sum_{\text{finite}} \frac{\xi_i}{d_i} \cdot f(x_i) \in J \Rightarrow \frac{a}{b} \in J$.

This proves that $I^e \subseteq J$. Hence $I^e = J$. 
\[ 2 \text{ Claim: } \quad I^{ec} = \bigcup_{s \in S} (I : s). \]

Let \( x \in I^{ec} = \overline{f}(I^e) \), where \( f : A \to \overline{s} A \)
given by \( f(a) = \frac{a}{y} \).

Hence \( f(x) \in I^{e} = \overline{s}(I) \). This gives \( f(x) = \frac{a}{y} \) for some \( a \in I \) and \( y \in S \).

But \( f(x) = \frac{1}{y} \). Hence \( x = \frac{a}{y} \). This gives \( (a \cdot 1 - axy)t = 0 \) for some \( t \in S \).

i.e. \( a + t = ayt \) where \( a \in I \).

\[ \Rightarrow \quad ayt \in I. \]

Thus \( x \in I : s' \). Thus \( I^{ec} \subseteq \bigcup_{s \in S} (I : s) \).
Conversely, assume that \( x \in U \cup I : s \).

Hence \( x \in I : t \) for some \( t \in S \). This gives \( x t \in I \). From the first part to show \( x \in I^{ec} \), it is enough to show that
\[
\frac{x}{1} = f(x) = \frac{a}{b}, \text{ where } a \in I \& b \in S.
\]

It is clear that \( \frac{x}{1} = \frac{x t}{t} = \frac{a}{b}, \text{ where } a = x t \text{ if } b = t \in S \). Thus \( x \in I^{ec} \).

Thus \( U \cup I : s \subseteq I^{ec} \).
In particular, \[ I^c = (1) = s^+ A \]
\[ \iff \quad I^{ec} = \left( s^+ A \right)^c = A = (1) \text{ unity of } A \]
\[ \iff \quad UI \cdot s = I^{ec} = (1) \exists 1 \]
\[ s \in S \]
\[ \iff \quad 1 \in I \cdot s \text{ for some } s \in S. \]
\[ \iff \quad s \in I \quad \text{ implies } \]
\[ \iff \quad I \cap S \neq \emptyset. \]
2. Let $I$ be a contracted ideal.

Then $I = I^e c (\text{Prop. 1.17})$

$= \bigcup I : S$.

**Claim:** No element of $S$ is a zero-divisor in $A/I$.

Suppose on the contrary that for some $x \in S$,

$(x + I)$ is a zero-divisor in $A/I$.

Hence $\exists y + I \neq I : \ (x + I)(y + I) = I$.

i.e. $xy + I = I$, i.e. $xy \in I$.

i.e. $y \in I : x \in \bigcup I : S = I^e c = I$.

This proves that $\{y \in I\}$, a contradiction. Thus the claim.
Conversely, assume that whenever 

\[(x + I)(y + I) = I \text{ for some } x \in S \Rightarrow y \in I.\]

**Claim:** \[I = I^e c.\]

Clearly, \[I \subseteq I^e c.\]

Let \[x \in I^e c = \bigcup_{I \in S} I^I.\] Then \[x \in I^I t \text{ for some } t \in S.\]

Some \(t \in S, i.e., x \cdot t \in I \text{ for } t \in S.\) i.e. \((x + I)(t + I) = x \cdot t + I = I \text{ } \& \text{ } t \in S\)

By the hypothesis, \(x \in I.\) This proves that \[\sqrt{I^e c} = I.\]
\[ f: A \to B, \text{ where } A = \{ p \in \text{spec}(A) \mid p \not\mid s = f \} \]

and \[ B = \{ q \in \text{spec}(S^{-1}A) \}. \]

**Claim:** \( f \) is bijective.

Let \( p \) be a prime ideal of \( A \) s.t. \( p \not\mid s \), i.e. \( p \not\subset A \). We prove that \( S^{-1}p \in B \), i.e. \( S^{-1}(p) \) is a prime ideal in \( S^{-1}A \).

Clearly, \( S^{-1}p \) is a proper ideal of \( S^{-1}A \).

Let \( \frac{a}{s}, \frac{b}{s} \in S^{-1}p. \) Then \( \frac{ab}{ss'} \in S^{-1}p. \)
Hence \( \frac{\text{ab}}{s} = \frac{x}{y} \) for \( x \in P \) \& \( y \in S \).

This gives \((\text{aby} - ss'x)t = 0\) for some \( t \leq S\).

\[
\Rightarrow \quad \text{aby}t = ss'xt \in P \quad (\text{as } x \in P).
\]

Since \( P \) is prime, we have \( \text{ab} \in P \)

as \( y, t \leq S \) and \( S \) is m.c.s \( \Rightarrow \text{yt} \leq S \) \& \( \text{pt} \in S \).

By primeness of \( P \) we have a coprime \( P \).

Hence \( \frac{a}{s} \in \bar{s}'p \) or \( \frac{b}{s} \in \bar{s}'p \).

Thus \( \bar{s}'p \in B \), i.e. \( \bar{s}'p \) is a prime ideal in \( \bar{S} \).
Let $J$ be a prime ideal in $\mathcal{S}A$. Then $J = \mathcal{S}^1(I)$ (as every ideal in $\mathcal{S}A$ is extended).

Where $I = \{ a \in A \mid \frac{a}{1} \in J \}$.

**Claim:** $I$ is a prime ideal of $A$ with $I \cap S = \emptyset$.

Clearly, $I$ is a proper ideal of $A$.

To prove $I$ is prime, assume that $ab \in I$. Then $\frac{ab}{1} = \frac{a}{1} \cdot \frac{b}{1} \in \mathcal{S}^1(I) = J$.

Being $J$ prime, $\frac{a}{1} \in \mathcal{S}^1(I)$ or $\frac{b}{1} \in \mathcal{S}^1(I)$. 
Hence either \( a \in I \) or \( b \in I \). Thus \( I \) is a prime ideal of \( A \).

If \( I \cap S \neq \emptyset \), then by previous result \( I^e = (1) = \Sigma A \).

\( \therefore J = \Sigma (I) = I^e = (1) \), a contradiction to properness of \( I \) (as \( J \) is prime).

\[ \Rightarrow I \cap S = \emptyset \Rightarrow f \text{ is onto.} \]

**Claim:** \( f \) is 1-1.

Let \( \Sigma^1(P_1) = \Sigma^1(P_2) \) for some \( P_1, P_2 \in \text{Spec}(A) \) \& \( P_1, P_2 \not\approx S = \emptyset = P_2 \cap S \).
\[ \Rightarrow f^{-1}(S_{P_1}) = f^{-1}(S_{P_2}) \]
\[ \Rightarrow P_{\text{ec}} = P_{\text{ec}} \]
\[ \Rightarrow P_1 = P_2 \]

Thus \( f \) is 1-1.

This proves that \( f \) is bijective.
v) The operation $S^{-1}$ commutes with formation of finite sums, products, intersections and radicals.

**Proof:** \( \dot{\circ} \quad \overline{S}(I+J) = \overline{S}(I) + \overline{S}(J) \)

Let \( x \in \overline{S}(I+J) \). Then

\[
\begin{align*}
  x &= \frac{i+j}{8} \quad \text{for } i \in I, j \in J \quad \forall \quad 8 \in S,
  \\
  &= \frac{i}{8} + \frac{j}{8} \quad \in \overline{S}(I) + \overline{S}(J)
\end{align*}
\]

Conversely, let \( t \in \overline{S}(I) + \overline{S}(J) \).

Then \( t = \frac{x+y}{8} \quad \text{for } x \in I, y \in J \quad \forall \quad 8 \in S. \)
Then \( t = xs' + ys' \) \\
\[ \frac{xs'}{ys'} \] (clearly), \\
\( ss' \in S \), \( x \in I \) and \( ys' \in J \). \\
Hence \( t \in \overline{S}(I + J) \).

This proves that \( \overline{S}(I + J) = \overline{S}(I) \cap \overline{S}(J) \). \\

Similarly, \( \overline{S}(I \cap J) = \overline{S}(I) \cup \overline{S}(J) \). \\
\( 4 \overline{S}(2 \triangle I) = 2 \overline{S}(4 \triangle I) \). (Exercise).
Let \( \varphi : \mathbb{Z} \to \mathbb{Z} [i] \) be a ring homomorphism. We have already noted that \((5)\) is prime in \(\mathbb{Z}\) whereas \((5)\mathbb{C}^2\) is not prime in \(\mathbb{Z} [i] \).

Consider \( S = \mathbb{Z} \), \(3\), \(3^2\), \(\ldots\). Clearly, \( S \) is mcs. Consider a homomorphism \( f : \mathbb{Z} \to S' \mathbb{Z} \) then

\[
(5) = \varphi^{-1}(5) = \left\{ \frac{5k}{3^j} \mid k \in \mathbb{Z}, \ j \in \mathbb{N}_0 \right\}.
\]

Here \((5)\) is a prime ideal in \(S' \mathbb{Z}\) as \((5) \cap S = \phi\). Thus, we are not able to find a mcs \( S \) s.t. \( \frac{1}{S} \mathbb{Z} = \mathbb{Z} [i] \).
Corollary 3.12. If $\mathfrak{N}$ is the nilradical of $A$, the nilradical of $S^{-1}A$ is $S^{-1}\mathfrak{N}$. ■

Corollary 3.13. If $\mathfrak{p}$ is a prime ideal of $A$, the prime ideals of the local ring $A_\mathfrak{p}$ are in one-to-one correspondence with the prime ideals of $A$ contained in $\mathfrak{p}$.

Proof. Take $S = A - \mathfrak{p}$ in (3.11) (iv). ■

Proof: follows from the fact that

$$S^{-1}(\bigcap P_i) = \bigcap S^{-1}(P_i).$$

Proof: follows from Proposition 3.11.

Ex.: Let $S$ be a mcs of $A$ and consider a map $f: A \rightarrow B$ a hom. Prove that $f(S) = \mathfrak{s}$ is a mcs of $B$. 
Proof: Let \( x, y \in \overline{S} = f(S) \).

Then \( x = f(s_1) \) and \( y = f(s_2) \) for some \( s_1, s_2 \in S \). Hence \( xy = f(s_1)f(s_2) = f(s_1s_2) = f(s) \). Clearly \( s \in S \).

Thus \( f(S) \) is a mcs.

**Proposition 3.16.** Let \( A \rightarrow B \) be a ring homomorphism and let \( p \) be a prime ideal of \( A \). Then \( p \) is the contraction of a prime ideal of \( B \) if and only if \( p^{eq} = q \).

Proof: Let \( \phi: A \rightarrow B \) be ring homo.

and \( p \) be a prime ideal. Then \( A/p \) is a mcs. Let \( S = \phi(A-p) \).
let \( \mathfrak{p}^e \) be the extension ideal of \( \mathfrak{p} \)
in \( B \) under \( \phi \), i.e. \( \langle \phi(p) \rangle = \mathfrak{p}^e \).
Then \( \mathfrak{p}^e \mathfrak{p}^e = \phi^e \). For this, let \( \mathfrak{b} \in \mathfrak{p}^e \).

Since \( \mathfrak{b} \mathfrak{c} = \phi(A \mathfrak{p}) \), we have \( \mathfrak{b} = \phi(a) \)
for some \( a \in \mathfrak{p} \). But then

\[
\phi^{-1}(b) = a \in \phi^{-1}(\mathfrak{p}^e) \quad (as \ \mathfrak{b} \in \mathfrak{p}^e)
= \mathfrak{p}^e = \mathfrak{p}, \ \text{a contradiction.}
\]

Let \( f: B \to \hat{\mathfrak{p}}^e B \) a natural homo.
defined by \( f(b) = \frac{b}{1} \).
Consider the map \( A \xrightarrow{f} B \xrightarrow{\overline{f}} \overline{S} \mathbb{B} \).

Now, consider the extension of \( pe \) in \( \overline{S} \mathbb{B} \), that is, the ideal generated by \( f(pe) \) in \( \overline{S} \mathbb{B} \).

**Claim:** The ideal generated by \( f(pe) \) in \( \overline{S} \mathbb{B} \) is proper.

Suppose on the contrary that

\[
\langle f(pe) \rangle = (1). \quad \text{(note that 1 of } \overline{S} \mathbb{B})
\]

But

\[
\frac{1}{1} = \sum_{i \text{ finite}} \frac{b_i}{\bar{s}_i} f(x_i) \quad \text{where } \frac{b_i}{\bar{s}_i} \in \overline{S} \mathbb{B}
\]

and \( x_i \in pe \).
\[ \Rightarrow \quad \frac{1}{1} = \sum_{\text{finite}}^{\infty} \frac{b_i}{s_i} \cdot \frac{x_i}{1} \quad (\text{as } f(x_i) = x_i^\perp) \]

\[ = \sum_{\text{finite}}^{\infty} \frac{b_i x_i}{s_i} \]

\[ \frac{1}{1} = \frac{x}{s}, \quad \text{where } x \in P^e \text{ and } s \in S. \]

\[ \Rightarrow \quad (x - s) s^\prime = 0 \text{ for some } s^\prime \in S. \]

\[ \Rightarrow \quad x s^\prime = s s^\prime. \quad \text{Clearly, } s s^\prime \in S \text{ and } x s^\prime \in P^e. \]

\[ \Rightarrow \quad x s^\prime \in P^e \cap S = \emptyset, \quad \text{a contradiction.} \]
Thus the ideal generated by \( \langle f(p^e) \rangle \) is a proper ideal of hence it is contained in a proper ideal, \( \mathfrak{J} \).

\[ \Rightarrow \mathfrak{J} \text{ is a prime ideal in } \mathfrak{S}^! \mathfrak{B}. \]

Clearly, \( \mathfrak{Q} = \mathfrak{S}^!(\mathfrak{J}) \) be a prime ideal of \( \mathfrak{B} \). (Note that inverse image of prime ideal is prime).

Thus \( \mathfrak{Q} = \mathfrak{J}^c \).

\[ \text{Claim: } \mathfrak{Q}^c = \mathfrak{P}. \]

Since \( f(p^e) \subseteq \langle f(p^e) \rangle \subseteq \mathfrak{J} \), we have
Now we prove that \( Q \cap S = \emptyset \).

i.e.

\[ f(J) \cap \phi(A) \cap P = \emptyset. \]

Let \((\mathfrak{g}^e)^\prime\) be the extension of \( Q \)
under \( f \) is a proper ideal of \( \mathfrak{S}^e B \).

Since \( p e^e \subseteq Q \), we have

\[ f(p e^e) \subseteq f(Q) = f(f^{-1}(e)) \subseteq J. \]

\( \mathfrak{g}^e = \langle f(Q) \rangle \subseteq \langle J \rangle = J \)
Since $J \neq (1)$, we have

$$\left(\mathcal{G}^{\epsilon}\right)' \neq (1).$$

We know the

assert that if $I^e \equiv (1) \iff \mathcal{I} \cap S = \emptyset$.

Hence

$$\mathcal{G} \cap S = \emptyset.$$  Thus the claim.

Now, we prove that $\mathcal{G}^e \cap (A \cup \mathcal{P}) = \emptyset$.

Let $a \in \mathcal{G} \cap (A \cup \mathcal{P})$. Then

$\mathcal{G}(a) \in \mathcal{Q}$ and $a \notin \mathcal{P}$. But then

$\mathcal{G}(a) \notin \mathcal{G}(A \cup \mathcal{P})$. 
Thus \( \phi(a) \in \mathcal{Q} \cap \phi(A \setminus \mathcal{P}) \).

\[
= \mathcal{Q} \cap \mathcal{S} = \phi \text{ (empty set)}
\]

ea contradiction.

Hence \( \phi^{-1}(\mathcal{Q}) \cap (A \setminus \mathcal{P}) = \emptyset \).

Now, we are ready to prove our final claim \( \phi^c = P \).

Since \( P^c \subseteq \mathcal{Q} \), we have

\[
P = P^c \subseteq \mathcal{Q}^c \subseteq \mathcal{Q}^c.
\]
Further, we have proved that
\[ \bar{g}(Q) \cap (\mathcal{A} \cup P) = \emptyset; \] we set
\[ \mathcal{Q} = \bar{g}(Q) \subseteq P. \] This together with
\[ Q \mathcal{Q} = Q \subseteq Q \] gives
\[ P \subseteq Q \mathcal{Q} \] gives
\[ P = Q \mathcal{Q}. \]

Conversely, assume that
\[ P = Q \mathcal{Q} \] for some prime ideal \( Q \) of \( B \).

Then
\[ P = Q \mathcal{Q} = Q = P. \] Thus \( P = Q \mathcal{Q} \)