Title: One Variable Separable Pfaffian Differential Equations

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One Variable Separable Pfaffian Differential Equations

Pfaffian Differential Equation:

Let $p(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ be functions of three variables $x,y,z$. Then differential equation (DE) of the form

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$$

Or

$$Pdx + Qdy + Rdz = 0$$

is called Pfaffian differential equation in three variables.

Let $\vec{X} = (P, Q, R)$ and $d\vec{r} = (dx, dy, dz)$

Then
\[ \vec{X} \cdot d\vec{r} = Pdx + Qdy + Rdz \]

Then equation (1) can be written as
\[ \vec{X} \cdot d\vec{r} = Pdx + Qdy + Rdz = 0 \]
i.e.
\[ \vec{X} \cdot d\vec{r} = 0 \]

Where \( \vec{X} = (P, Q, R) \) and \( d\vec{r} = (dx, dy, dz) \)

This is vector form of Pffafian DE.

**Theorem:** Let \( \vec{X} = (P, Q, R) \), where P,Q,R are functions of \( x, y, z \) and \( d\vec{r} = (dx, dy, dz) \). Then the Pffafian DE
\[ \vec{X} \cdot d\vec{r} = Pdx + Qdy + Rdz = 0 \]
i.e. \( \vec{X} \cdot d\vec{r} = 0 \) is integrable iff \( \vec{X} \cdot \text{curl} \vec{X} = 0 \)

**Method to obtain solutions of One Variable Separable Pffafian Differential Equations:**

Consider Pffafian DE in three variables \( x, y, z \) as
\[ P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0 \] \hspace{1cm} (1)
which is integrable.

Suppose this Pffafian DE (1) is one variable separable, let \( z \) variable be separable.

Then equation (1) becomes
\[ P_1(x, y)dx + Q_1(x, y)dy + R_1(z)dz = 0 \] \hspace{1cm} (2)

Now
\[
\text{curl} \vec{X} = \begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
P_1(x, y) & Q_1(x, y) & R_1(z)
\end{bmatrix}
\]

\[
= i \left( \frac{\partial R_1(z)}{\partial y} - \frac{\partial Q_1(x, y)}{\partial z} \right) - j \left( \frac{\partial R_1(z)}{\partial x} - \frac{\partial P_1(x, y)}{\partial z} \right) + k \left( \frac{\partial Q_1(x, y)}{\partial x} - \frac{\partial P_1(x, y)}{\partial y} \right)
\]
\[
\begin{align*}
\mathbf{i}(0 - 0) - \mathbf{j}(0 - 0) + \mathbf{k} \left( \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} \right) \\
= \mathbf{i}(0) + \mathbf{j}(0) + \mathbf{k} \left( \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} \right)
\end{align*}
\]

Now

\[
\mathbf{\bar{X}} \cdot \text{curl} \mathbf{\bar{X}} = P_1(0) + Q_1(0) + R_1 \left( \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} \right)
\]

\[
= R_1 \left( \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} \right)
\]

We know that the Pfaffian DE

\[
P_1(x, y)dx + Q_1(x, y)dy + R_1(z)dz = 0
\]

is integrable iff \( \mathbf{\bar{X}} \cdot \text{curl} \mathbf{\bar{X}} = 0 \).

As the given DE (1) is integrable, so we have

\[
\mathbf{\bar{X}} \cdot \text{curl} \mathbf{\bar{X}} = 0
\]

That is

\[
R_1 \left( \frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} \right) = 0
\]

\[
\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} = 0
\]

as \( R_1(z) \neq 0 \)

\[
\therefore \frac{\partial Q_1}{\partial x} = \frac{\partial P_1}{\partial y}
\]

Therefore the DE

\[
P_1(x, y)dx + Q_1(x, y)dy = 0
\]

is exact DE.

Then \( P_1(x, y)dx + Q_1(x, y)dy \) is complete derivative of some function of \( x, y \), let it be \( v(x, y) \)

i.e.

\[
P_1(x, y)dx + Q_1(x, y)dy = dv
\]
Therefore equation (1) becomes

\[ dv + R_1(z)dz = 0 \]

Integrating, we get

\[ v + \int R_1(z)dz = c \]

This is the general solution or complete integral or integral curves or primitives of given DE (1).

**Problems:**

Ex.1- Solve or show that the DE is integrable and solve it or find complete integral or find integral curves or primitives of

\[ x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0 \]

Sol.- The given DE

\[ x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0 \quad (1) \]

is Pfaffian DE of the form

\[ Pdx + Qdy + Rdz = 0 \]

with \( P = x(y^2 - a^2), Q = y(x^2 - z^2), R = -z(y^2 - a^2) \)

Let \( \vec{X} = (P, Q, R) = (x(y^2 - a^2), y(x^2 - z^2), -z(y^2 - a^2)) \)

and \( d\vec{r} = (dx, dy, dz) \)

Then the given DE (1) can be written as

\[ \vec{X} \cdot d\vec{r} = 0 \]

We know that the Pfaffian DE

\[ Pdx + Rdy + Rdz = 0 \]

is integrable iff \( \vec{X} \cdot curl\vec{X} = 0. \)

Now
\[
\text{curl}\vec{X} = \begin{bmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x(y^2 - a^2) & y(x^2 - z^2) & -z(y^2 - a^2)
\end{bmatrix}
\]

\[
= \vec{i}\left(\frac{\partial}{\partial y}[-z(y^2 - a^2)] - \frac{\partial}{\partial z}[y(x^2 - z^2)]\right) - \vec{j}\left(\frac{\partial}{\partial x}[-z(y^2 - a^2)] - \frac{\partial}{\partial z}[x(y^2 - a^2)]\right) + \vec{k}\left(\frac{\partial}{\partial x}[y(x^2 - z^2)] - \frac{\partial}{\partial y}[x(y^2 - a^2)]\right)
\]

\[
= i(-2yz + 2yz) - j(0 - 0) + k(2xy - 2xy)
\]

\[
= i(0) + j(0) + k(0)
\]

Now

\[
\overline{X} \cdot \text{curl}\overline{X} = x(y^2 - a^2)(0) + y(x^2 - z^2)(0) + (-z(y^2 - a^2))(0)
\]

\[
= 0 + 0 + 0
\]

\[
= 0
\]

Therefore the given DE (1) is integrable.

Hence

\[
x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0
\]

\[
y(x^2 - z^2)dy = z(y^2 - a^2)dz - x(y^2 - a^2)dx
\]

\[
y(x^2 - z^2)dy = (y^2 - a^2)(zd - xdx)
\]

\[
\frac{ydy}{y^2 - a^2} = \frac{zd - xdx}{x^2 - z^2}
\]

Therefore the DE (1) is one variable separable type and variable \(y\) is separated.

\[
\frac{ydy}{y^2 - a^2} = \frac{xdx - zdz}{x^2 - z^2}
\]

\[
\frac{1}{2} \frac{2ydy}{y^2 - a^2} = -\frac{1}{2} \frac{12xdx - 2zdz}{x^2 - z^2}
\]

\[
\frac{2ydy}{y^2 - a^2} = -\frac{2xdx - 2zdz}{x^2 - z^2}
\]
\[ d[\log(y^2 - a^2)] = -d[\log(x^2 - z^2)] \]

Integrating, we get

\[ \log(y^2 - a^2) = -\log(x^2 - z^2) + c \]

\[ \log(y^2 - a^2) + \log(x^2 - z^2) = c \]

\[ \log((y^2 - a^2)(x^2 - z^2)) = c \]

\[ (y^2 - a^2)(x^2 - z^2) = e^c \]

\[ (y^2 - a^2)(x^2 - z^2) = c_1 \]

Where \( c_1 = e^c \)

This is the general solution or complete integral or integral curves or primitives of given DE (1).

OR from equation (2) we can use another method as:

\[ \frac{ydy}{y^2 - a^2} = \frac{zdz - xdx}{x^2 - z^2} \quad (2) \]

or

\[ \frac{ydy}{y^2 - a^2} = \frac{z}{x^2 - z^2}dz - \frac{x}{x^2 - z^2}dx \]

Let

\[ M = \frac{z}{x^2 - z^2} \text{ and } N = -\frac{x}{x^2 - z^2} \]

\[ \frac{\partial M}{\partial x} = \frac{\partial}{\partial x} \left( \frac{z}{x^2 - z^2} \right) = \frac{(x^2 - z^2)(0) - z(2x)}{(x^2 - z^2)^2} = \frac{-2xz}{(x^2 - z^2)^2} \]

\[ \frac{\partial N}{\partial z} = \frac{\partial}{\partial z} \left( -\frac{x}{x^2 - z^2} \right) = \frac{(x^2 - z^2)(0) - x(-2z)}{(x^2 - z^2)^2} = -\frac{2xz}{(x^2 - z^2)^2} \]
\[ \frac{-2xz}{(x^2 - z^2)^2} \]

\[ \therefore \frac{\partial M}{\partial x} = \frac{\partial N}{\partial z} \]

\[ \therefore Mdz + Ndx = 0 \] is exact DE.

i.e. \( \frac{z}{x^2-z^2} \, dz - \frac{x}{x^2-z^2} \, dx \) is exact DE.

Therefore its solution is

\[ \int Mdz + \int (\text{Terms in } N \text{ independent of } z) \, dx = c \]

\[ \int \frac{z}{x^2-z^2} \, dz + \int 0 \, dx = c \]

\[ \frac{1}{-2} \int \frac{-2z}{x^2-z^2} \, dz = c \]

\[ -\frac{1}{2} \log(x^2 - z^2) = c \]

Therefore equation (2) becomes

\[ \frac{1}{2} \frac{2ydy}{y^2-a^2} = d \left[ -\frac{1}{2} \log(x^2 - z^2) \right] \]

\[ d \left[ \frac{1}{2} \log(y^2 - a^2) \right] = -d \left[ \frac{1}{2} \log(x^2 - z^2) \right] \]

Integrating, we get

\[ \frac{1}{2} \log(y^2 - a^2) = -\frac{1}{2} \log(x^2 - z^2) + c_2 \]

\[ \log(y^2 - a^2) + \log(x^2 - z^2) = c \]

Where \( c = 2c_2 \)

\[ \log((y^2 - a^2)(x^2 - z^2)) = c \]

\[ (y^2 - a^2)(x^2 - z^2) = e^c \]

\[ (y^2 - a^2)(x^2 - z^2) = c_1 \]

Where \( c_1 = e^c \)
This is the general solution or complete integral or integral curves or primitives of given DE (1).

Ex.2- Solve or show that the DE is integrable and solve it or find complete integral or find
integral curves or primitives of

\[ yzdx - zxdy - y^2dz = 0 \]

Sol.- The given DE

\[ yzdx - zxdy - y^2dz = 0 \]  \hspace{1cm} (1)

is Pfaffian DE of the form

\[ Pdx + Qdy + Rdz = 0 \]

with \( P = yz, Q = -zx, R = -y^2 \)

Let \( \mathbf{X} = (P, Q, R) = (yz, -zx, -y^2) \)

and \( d\mathbf{r} = (dx, dy, dz) \)

Then the given DE (1) can be written as

\[ \mathbf{X} \cdot d\mathbf{r} = 0 \]

We know that the Pfaffian DE

\[ Pdx + Rdy + Rdz = 0 \]

is integrable iff \( \mathbf{X} \cdot \text{curl} \mathbf{X} = 0 \).

Now

\[
\text{curl} \mathbf{X} = \begin{bmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
yz & -zx & -y^2
\end{bmatrix}
\]

\[
= \mathbf{i} \left( \frac{\partial(-y^2)}{\partial y} - \frac{\partial(-zx)}{\partial z} \right) - \mathbf{j} \left( \frac{\partial(-y^2)}{\partial x} - \frac{\partial(yz)}{\partial z} \right) + \mathbf{k} \left( \frac{\partial(-zx)}{\partial x} - \frac{\partial(yz)}{\partial y} \right)
\]

\[
= i(-2y + x) - j(0 - y) + k(-z - z)
\]

\[
= i(-2y + x) + j(y) + k(-2z)
\]
Now
\[ \vec{X} \cdot \text{curl} \vec{X} = yz(-2y + x) + (-zx)y + (-y^2)(-2z) \]
\[ = -2y^2z + xyz - xyz + 2y^2z \]
\[ = 0 \]

Therefore the given DE (1) is integrable.

Hence
\[ yzdx - xzdy - y^2dz = 0 \]

Dividing by \( z \), we get
\[ ydx - xdy = \frac{1}{z}y^2dz \]
\[ \frac{ydx - xdy}{y^2} = \frac{dz}{z} \]

Therefore the DE (1) is one variable separable type and variable \( z \) is separated.

\[ d \left( \frac{x}{y} \right) = \frac{dz}{z} \]

Integrating, we get
\[ \frac{x}{y} = \log z + c \]

This is the general solution or complete integral or integral curves or primitives of given DE (1).

Exercises:
Ex. Test whether the following DEs are integrable, if yes solve them or find their complete integrals

1. \( yzdx - xzdy - y^2dz = 0 \)
2. \( (y + z)dx + dy + dz = 0 \)
3. \( (xz + y^2)dx + (x^2 + y^2)dy + 3zdz = 0 \)
4. \( x(1 + z^2)dx + y(1 + z^2)dy + (x^2 + y^2)dz = 0 \)
5. \((y^2 + z^2)dx + yxdy + xzdz = 0\)
6. \(y^2zdx + zx^2dy - x^2y^2dz = 0\)
7. \(y(1 + z^2)dx - x(1 + z^2)dy + (x^2 + y^2)dz = 0\)

References:

1. Ian Sneddon, Elements Partial Differential Equations