Title: Pffafian Differential Equation

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Pffafian Differential Equation

Let \( p(x, y, z) \), \( Q(x, y, z) \), \( R(x, y, z) \) be functions of three variables \( x, y, z \). Then differential equation (DE) of the form

\[
P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0
\]

Or

\[
Pdx + Qdy + Rdz = 0 \tag{1}
\]

is called Pffafian differential equation in three variables.

Let \( \vec{X} = (P, Q, R) \) and \( d\vec{r} = (dx, dy, dz) \)

Then

\[
\vec{X} \cdot d\vec{r} = Pdx + Qdy + Rdz
\]
Then equation (1) can be written as

\[ \overline{X} \cdot d \overline{r} = Pdx + Qdy + Rdz = 0 \]

i.e.

\[ \overline{X} \cdot d \overline{r} = 0 \]

Where \( \overline{X} = (P, Q, R) \) and \( d \overline{r} = (dx, dy, dz) \)

This is vector form of Pfaffian DE.

Ex. Define Pfaffian DE in three variables and vector form of Pfaffian DE.

Theorem (1): If \( \overline{X} \) is vector and \( \mu \) is function of \( x, y, z \) then \( \overline{X} \cdot \text{curl}\overline{X} = 0 \) iff \( \mu\overline{X} \cdot \text{curl}(\mu\overline{X}) = 0 \)

Proof: Part –I : Necessary condition

Assume that

\[ \overline{X} \cdot \text{curl}\overline{X} = 0 \]

Now

\[ \text{curl}(\mu\overline{X}) = \nabla \times (\mu\overline{X}) \]

\[ = \nabla \mu \times \overline{X} - \mu \nabla \times \overline{X} \]

As

\[ \text{curl}(\emptyset \overline{v}) = \nabla \emptyset \times \overline{v} - \emptyset \nabla \times \overline{v} \]

Now

\[ \mu\overline{X} \cdot \text{curl}(\mu\overline{X}) = \mu\overline{X} \cdot [\nabla \mu \times \overline{X} - \mu \nabla \times \overline{X}] \]

\[ = \mu\overline{X} \cdot \nabla \mu \times \overline{X} - \mu\overline{X} \cdot \mu \nabla \times \overline{X} \]

\[ = \mu(\overline{X} \cdot \nabla \mu \times \overline{X}) - \mu^2(\overline{X} \cdot \nabla \times \overline{X}) \]

\[ = \mu(0) - \mu^2(0) \]

by assumption and if two vectors are similar in scalar triple product then it equal to zero.

\[ \therefore \mu\overline{X} \cdot \text{curl}(\mu\overline{X}) = 0 \]
Part-II : Sufficient condition

Assume that

\[ \mu \bar{X} \cdot \text{curl}(\mu \bar{X}) = 0 \]

\[ \mu \bar{X} \cdot (\nabla \mu \times \bar{X} - \mu \nabla \times \bar{X}) = 0 \]

\[ \mu \bar{X} \cdot \nabla \mu \times \bar{X} - \mu \bar{X} \cdot \mu \nabla \times \bar{X} = 0 \]

\[ \mu (\bar{X} \cdot \nabla \mu \times \bar{X}) - \mu^2 (\bar{X} \cdot \nabla \times \bar{X}) = 0 \]

\[ \mu(0) - \mu^2 (\bar{X} \cdot \nabla \times \bar{X}) = 0 \]

as if two vectors are similar in scalar triple product then it equal to zero

\[ \therefore \mu^2 (\bar{X} \cdot \nabla \times \bar{X}) = 0 \]

\[ \therefore \bar{X} \cdot \nabla \times \bar{X} = 0 \text{ as } \mu^2 \neq 0 \]

Hence theorem.

Theorem (2): Let \( \bar{X} = (P, Q, R) \), where \( P, Q, R \) are functions of \( x, y, z \) and

\[ d\bar{r} = (dx, dy, dz) \]. Then the Pfaffian DE

\[ \bar{X} \cdot d\bar{r} = Pdx + Qdy + Rdz = 0 \]

i.e. \( \bar{X} \cdot d\bar{r} = 0 \) is integrable iff \( \bar{X} \cdot \text{curl}\bar{X} = 0 \)

Proof : Part –I : Necessary condition

Let \( \bar{X} = (P, Q, R) \), where \( P, Q, R \) are functions of \( x, y, z \) and \( d\bar{r} = (dx, dy, dz) \).

Assume that the Pfaffian DE

\[ \bar{X} \cdot d\bar{r} = Pdx + Qdy + Rdz = 0 \]  \( \text{(1)} \)

i.e. \( \bar{X} \cdot d\bar{r} = 0 \) is integrable.

Hence it has solution, let it be

\[ \emptyset(x, y, z) = c_1 \]

If equation (1) is exact DE, then we have
\[ \frac{\partial \phi}{\partial x} = P, \frac{\partial \phi}{\partial y} = Q, \frac{\partial \phi}{\partial z} = R \]

\[ \therefore \vec{X} = (P, Q, R) \]

\[ \vec{X} = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \nabla \phi \]

Now

\[ \text{curl} \vec{X} = \nabla \times \vec{X} = \nabla \times \nabla \phi = 0 \]

as \( \text{curl} (\text{grad} \phi) = 0 \) for any scalar function \( \phi \)

\[ \vec{X} \cdot \text{curl} \vec{X} = \vec{X} \cdot 0 = 0 \]

\[ \therefore \vec{X} \cdot \text{curl} \vec{X} = 0 \]

If equation (1) is not exact DE, then it has integrating factor, let it be \( \mu = \mu(x, y, z) \).

Therefore

\[ \mu P \, dx + \mu Q \, dy + \mu R \, dz = 0 \]

becomes exact.

Let its solution be

\[ \psi(x, y, z) = c_2 \]

Then we have

\[ \frac{\partial \psi}{\partial x} = \mu P, \frac{\partial \psi}{\partial y} = \mu Q, \frac{\partial \psi}{\partial z} = \mu R \]

\[ \therefore \mu \vec{X} = (\mu P, \mu Q, \mu R) \]

\[ \vec{X} = \left( \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z} \right) = \nabla \psi \]
Now

\[
\text{curl}(\mu \vec{X}) = \nabla \times \mu \vec{X} = \nabla \times \nabla \psi = 0
\]

as \(\text{curl}(\text{grad}\phi) = 0\) for any scalar function \(\phi\)

\[
\mu \vec{X} \cdot \text{curl}(\mu \vec{X}) = \mu \vec{X} \cdot \vec{0} = 0
\]

\[
\mu \vec{X} \cdot \text{curl}(\mu \vec{X}) = 0
\]

Therefore by above theorem (1), we have

\[
\vec{X} \cdot \text{curl} \vec{X} = 0
\]

Part II: Sufficient condition

Assume that

\[
\vec{X} \cdot \text{curl} \vec{X} = 0
\]

To prove: \(\vec{X} \cdot d\vec{r} = 0\) is integrable.

That is

\[
Pdx + Qdy + Rdz = 0
\]

is integrable.

Treating

\[
z = \text{constant}
\]

\[
\therefore \ dz = 0
\]

Therefore equation (1) becomes

\[
Pdx + Qdy = 0
\]

(2)

This is Pfaffian DE in two variable and so is integrable and so it has integrating factor, let it be \(\mu = \mu(x, y, z)\).

Therefore

\[
\mu Pdx + \mu Qdy = 0
\]
becomes exact.

Let its solution be

$$\psi(x, y, z) = c_1$$

Then we have

$$\frac{\partial \psi}{\partial x} = \mu P, \quad \frac{\partial \psi}{\partial y} = \mu Q$$

$$\frac{1}{\mu} \frac{\partial \psi}{\partial x} = P, \quad \frac{1}{\mu} \frac{\partial \psi}{\partial y} = Q$$

Equation (1) becomes

$$\frac{1}{\mu} \frac{\partial \psi}{\partial x} dx + \frac{1}{\mu} \frac{\partial \psi}{\partial y} dy + Rdz = 0$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \mu Rdz = 0$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz + \mu Rdz - \frac{\partial \psi}{\partial z} dz = 0$$

$$\nabla \psi \cdot d\bar{r} + \left( \mu R - \frac{\partial \psi}{\partial z} \right) dz = 0$$  \hspace{1cm} (3)

$$\mu R - \frac{\partial \psi}{\partial z} = \eta$$

$$\therefore \mu R = \eta + \frac{\partial \psi}{\partial z}$$

So equation (3) becomes

$$\nabla \psi \cdot d\bar{r} + \eta dz = 0$$  \hspace{1cm} (4)

And

$$\therefore \mu \vec{X} = (\mu P, \mu Q, \mu R)$$

$$= \left( \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \eta + \frac{\partial \psi}{\partial z} \right)$$
\[
\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z}) + (0, 0, \eta) = \nabla \psi + \eta \vec{k}
\]

Now

\[
\text{curl}(\mu \vec{X}) = \nabla \times (\nabla \psi + \eta \vec{k})
\]

\[
= \nabla \times \nabla \psi + \nabla \times \eta \vec{k}
\]

\[
= \vec{0} + \nabla \times \eta \vec{k}
\]

\[
= \nabla \times \eta \vec{k}
\]

\[
= \left[ \begin{array}{ccc}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & 0 & \eta
\end{array} \right]
\]

\[
= \vec{i} \left( \frac{\partial \eta}{\partial y} - 0 \right) - \vec{j} \left( \frac{\partial \eta}{\partial x} - 0 \right) + \vec{k} (0 - 0)
\]

\[
= \vec{i} \left( \frac{\partial \eta}{\partial y} \right) - \vec{j} \left( \frac{\partial \eta}{\partial x} \right) + \vec{k} (0)
\]

\[
\mu \vec{X} \cdot \text{curl}(\mu \vec{X}) = \frac{\partial \psi \partial \eta}{\partial x \partial y} - \frac{\partial \psi \partial \eta}{\partial y \partial x} + 0
\]

\[
= \frac{\partial (\psi, \eta)}{\partial (x, y)}
\]

By assumption

\[
\vec{X} \cdot \text{curl}(\vec{X}) = 0
\]

\[
\Rightarrow \mu \vec{X} \cdot \text{curl}(\mu \vec{X}) = 0
\]

by above theorem

\[
\Rightarrow \frac{\partial (\psi, \eta)}{\partial (x, y)} = 0
\]

Therefore there exists a relation between \( \psi \) and \( \eta \), not involving \( x \) or \( y \) explicitly.

Let it be \( \Phi(\psi, \eta) = 0 \).
And hence from equation (4), \( \eta \) is the function of \( \psi \) and \( z \) alone.

i.e. \( \eta = \eta(\psi, z) \)

Hence from equation (4), we have

\[
d\psi + \eta(\psi, z)dz = 0
\]

Therefore it has solution, let it be

\[
F(\psi, z) = c_2
\]

Using this in equation \( U(x, y, z) = c_3 \) and this is the solution of equation (1).

Hence equation (1) is integrable.

Hence theorem.

Ex.1- Solve or show that the DE is integrable and solve it or find complete integral or find integral curves or primitives of

\[
ydx + xdy + zdz = 0
\]

Sol.- The given DE

\[
ydx + xdy + zdz = 0 \quad (1)
\]

is Pfaffian DE of the form

\[
Pdx + Qdy + Rdz = 0
\]

with \( P = y, Q = x, R = z \)

Let \( \vec{X} = (P, Q, R) = (y, x, z) \)

and \( d\vec{r} = (dx, dy, dz) \)

Then the given DE (1) can be written as

\[
\vec{X} \cdot d\vec{r} = 0
\]

We know that the Pfaffian DE

\[
Pdx + Rdy + Rdz = 0
\]

is integrable iff \( \vec{X} \cdot \text{curl} \vec{X} = 0 \).

Now
\[ \text{curl} \vec{X} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \]

\[ = i \left( \frac{\partial z}{\partial y} - \frac{\partial x}{\partial z} \right) - j \left( \frac{\partial z}{\partial x} - \frac{\partial y}{\partial z} \right) + k \left( \frac{\partial x}{\partial y} - \frac{\partial y}{\partial x} \right) \]

\[ = i(0 - 0) - j(0 - 0) + k(1 - 1) \]

\[ = i(0) + j(0) + k(0) \]

Now

\[ \vec{X} \cdot \text{curl} \vec{X} = y(0) + x(0) + z(0) \]

\[ = 0 \]

Therefore the given DE (1) is integrable.

Hence

\[ ydx + xdy + zdz = 0 \]

\[ d(yx) + zdz = 0 \]

Integrating, we get

\[ yx + \frac{z^2}{2} = c \]

This general solution or complete integral or integral curves or primitives of given DE (1).