UNIT 01 - Prerequisites.

- Countable and Uncountable Sets:

A set $A$ is said to be countable if there exists a bijective map $f: \mathbb{Z}^+ \to A$.

A set $A$ is said to be infinite if it is not finite. It is said to be countably infinite if there exists a bijective correspondence $f: A \to \mathbb{Z}^+$.

Example: The set $\mathbb{Z}$ of all integers is countably infinite.

Define a map $f: \mathbb{Z} \to \mathbb{Z}^+$ by

$$f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2n+1 & \text{if } n \leq 0 \end{cases}$$

is a bijection.

Definition: A set is said to be countable if it is either finite or countably infinite. A set that is not countable is said to be uncountable.

2. Show that the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Solution: $\mathbb{Z} \times \mathbb{Z} = \{(x, y) | x, y \in \mathbb{Z}^+\}$. If we plot the elements...
of \( \mathbb{Z} \times \mathbb{Z}^+ \) as a points in the first quadrant of xy plane.

Define function \( f: \mathbb{Z} \times \mathbb{Z}^+ \rightarrow \mathbb{A} \) as \( f(x,y) = (x+y-1, y) \)

and another function \( g: \mathbb{1} \rightarrow \mathbb{Z}^+ \) as \( g(x,y) = \frac{1}{2} (x-1)x+y \)

Thus clearly function \( f \) and \( g \) are bijections and

Composition of two bijections is also bijection.

Saf! \( \mathbb{Z} \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) is a bijection.

\( \therefore \) \( \mathbb{Z} \times \mathbb{Z}^+ \) is countably infinite.

Consider the product set \( \mathbb{Z} \times \mathbb{Z}^+ \) as exhibited below.

The set \( \mathbb{Z} \times \mathbb{Z}^+ \) can be written in an infinite sequence of diagonal elements as follows (11), (21), (31, 13), (41), ..., and,

\( \therefore \) \( \mathbb{Z} \times \mathbb{Z}^+ \) is countably infinite.
Corollary: If $A$ is finite, there is no bijection of $A$ with a proper subset of itself.

Proof: Assume that $B$ is a proper subset of $A$ and $f: A \rightarrow B$ is a bijection. By assumption, there is a bijection $g: A \rightarrow \{1, 2, \ldots, n\}$ for some $n$. The composite $gof$ is then a bijection of $B$ with $\{1, 2, \ldots, n\}$. This is a contradiction to the existence of a bijection of $B$ with itself.

Corollary: $\mathbb{Z}^+$ is not finite.

Proof: The function $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ defined by $f(n) = n + 1$ is a bijection of $\mathbb{Z}^+$ with a proper subset of itself.

Corollary: The cardinality of a finite set $A$ is uniquely determined by $A$.

Proof: Let $m, n$. Suppose there are bijections $f: A \rightarrow \{1, 2, \ldots, m\}$ and $g: A \rightarrow \{1, 2, \ldots, n\}$. Then the composite $gof: A \rightarrow \{1, 2, \ldots, m\}$ is a bijection of the finite set of $\{1, 2, \ldots, m\}$ with a proper subset of itself. This is a contradiction to the uniqueness of the cardinality of $A$.

Corollary: If $B$ is a subset of the finite set $A$, then $B$ is finite. If $B$ is a proper subset of $A$, then the cardinality of $B$ is less than the cardinality of $A$.
Countable and Uncountable sets:

The following are equivalent:

1. \( B \) is countable
2. There is a surjective function \( f: \mathbb{Z}_+ \to B \)
3. There is an injective function \( g: B \to \mathbb{Z}_+ \)

**Proof:**

1. \( \implies \) 2.

Suppose that \( B \) is countable.

If \( B \) is countably infinite, there is a bijection \( f: \mathbb{N} \to B \)
by definition and we are through.

If \( B \) is finite, there is a bijection \( h: \{1, 2, \ldots, n\} \to B \)
for some \( n \in \mathbb{N} \).

Define \( k: \mathbb{N} \to B \) by \( k(i) = h(i) \).

We can extend \( k \) to a surjective function \( f: \mathbb{Z}_+ \to B \) by defining

\[
f(0) = \begin{cases} k(1) & \text{for } i = 1 \\ k(i) & \text{for } i > 1 \end{cases}
\]

2. \( \implies \) 2.

Let \( B \) be non-empty set such that \( f: \mathbb{Z}_+ \to B \) is a surjective map.

**Claim:** To show that \( g: B \to \mathbb{Z}_+ \) is an injective map.

Define \( g: B \to \mathbb{Z}_+ \) by

\[
g(b) = \text{smallest element of } f^{-1}(b), \ b \in B
\]

Then by construction \( g \) is well defined as smallest positive integer exist in \( \mathbb{Z}_+ \) with \( g(b) = g(b') \).

\( \therefore \) smallest element of \( f^{-1}(b) \) is smallest element of \( f(b) \) if \( b = b' \).

Hence \( g: B \to \mathbb{Z}_+ \) is an injective map.

\[ (2) \implies (3) \text{ holds.} \]

(3) \( \implies \) (2).

Let \( B \) be a non-empty set such that \( g: B \to \mathbb{Z}_+ \) is an injective map.

**Claim:** There is one-to-one correspondence between \( B \) and \( \mathbb{Z}_+ \)

It is sufficient to show that every subset of \( \mathbb{Z}_+ \) is countable.
Let $A$ be any subset of $\mathbb{N}$.

Case (i): If $A$ is finite, hence it is countable.
- $B$ is countable.

Case (ii): If $A$ is infinite subset of $\mathbb{N}$.
- As $\mathbb{N}$ is an ordered set,
- every element of $\mathbb{N}$ can be compared, and hence in $A$ also.
- Every element of $A$ is countable.
- $B$ is countable.

$\Rightarrow$ $B = \{1\}$ holds.

**TM:** Show that a subset of a countable set is countable.

**Proof:** Let $A \subseteq B$ be any set such that $B$ is countable.
- If $f : B \rightarrow \mathbb{N}$ an injective function.
- Now $A \subseteq B$.
- Define $h : A \rightarrow \mathbb{N}$ by $h(a) = f(a)$, $a \in A$.
- Then $h = f|A$ such that $f$ is injunctive map.
- As $h$ is also injective map from $A$ to $\mathbb{N}$.
- If $A$ is countable. (by previous, TM)

Ex: If $A$ is an infinite subset of $\mathbb{N}$ then $A$ is also countable.

**Proof:** Define $f : A \rightarrow \mathbb{N}$ by
- $f(a) = \begin{cases} 2n & n \geq 0 \\ -2n+1 & n < 0 \end{cases}$
They clearly $f$ is one-one and onto from $\mathbb{Z} \rightarrow \mathbb{Z^+}$.

Given $A$ is infinite subset of $\mathbb{Z}^+$, then $f: A \rightarrow \mathbb{Z}$

Define $h: A \rightarrow \mathbb{Z^+}$ by $\{h(a) = 5(\text{a})\}$.

Thus $h \circ f$ is a one-one map from $A$ to a subset of $\mathbb{Z^+}$.

By above theorem, $A$ is also countable.

Ex.: Show that $\mathbb{Q}$ is countable.

**Solution:** For each $n \in \mathbb{Z^+}$ defined $q_n$ as:

$$q_n = \left\{ \frac{1}{n} \right\} \cup \{1, -1/2, 1/2, -1, 1, -2, 2, -1/3, 1/3, -2/3, 2/3, \ldots \}$$

Define $f: \mathbb{Z} \rightarrow \mathbb{Q}$ as:

$$f(k) = \begin{cases} \frac{k}{2} & \text{if } k \in \mathbb{Z}, \text{ k even} \\ k^2 & \text{if } k \in \mathbb{Z}, \text{ k odd} \\ 0 & \text{if } k = 0 \end{cases}$$

Then $f$ is one-one and onto.

Therefore $\mathbb{Q}$ is countable.

**Corollary:** The set $\mathbb{Z^+} \times \mathbb{Z^+}$ is countably infinite.

**Proof:** By theorem (8), it is sufficient to construct an injective map $f: \mathbb{Z^+} \times \mathbb{Z^+} \rightarrow \mathbb{Z^+}$

We define $f$ by the equation:

$$f((p, q)) = 2^p 3^q$$

For injective:

Suppose that $f(p, q) = f(r, s) \Rightarrow 2^p 3^q = 2^r 3^s \Rightarrow 2^p = 2^r \land 3^q = 3^s$. If $p = r$ then $3^q = 3^s \Rightarrow q = s$.

If $p < r$ then $2^p = 2^r \Rightarrow p = r$, impossible.

If $p > r$ then $3^q = 3^s \Rightarrow q = s$, impossible.

As $p = r$, then $3^q = 3^s \Rightarrow q = s$, impossible.

Therefore $f$ is injective.

As $n = p$, $3^m = 3^q$
Then if \( m | q \cdot n \),
\[
\Rightarrow \quad 1 = q \cdot m
\]
This is not possible.
\[
\Rightarrow \quad m | q \quad \text{simply because } q \cdot m
\]
\[
\Rightarrow \quad m = q.
\]
\[
\Rightarrow \quad n = p \quad \text{since } m | q
\]
\[
(\text{since } m | q)
\]
\[
\Rightarrow \quad f \text{ is injective.
}
\]
\[
\mathbb{Z}^+ \times \mathbb{Z}^+ \quad \text{is Countably infinite.}
\]

Ex. Show that \( \mathbb{Q}_+ \) of positive rational numbers is Countably infinite.

So \( \exists \) \( f: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Q}_+ \) by the
\[
f(nm) = m/n
\]
Because \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) is Countable, there is a surjection
\[f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+.
\]

Then the composite \( f \circ f: \mathbb{Z}^+ \rightarrow \mathbb{Q}_+ \) is a surjection.

By \( \text{Thm. (b)} \),
\[
\mathbb{Q}_+ \quad \text{is Countable}
\]
\[
\mathbb{Q}_+ \quad \text{is Countably infinite because } \mathbb{Z}^+ \text{ is infinite, & it contained in } \mathbb{Q}_+.
\]

Thm. A countable union of countable sets is countable.

Proof:\ Let \( \{A_n\}_{n \in \mathbb{N}} \) be any indexed family of countable sets, where the index set \( J \). Either \( J = \mathbb{N} \) or \( J = \mathbb{Z}^+ \).

Assume that each \( A_n \).

Claim: \( \cup A_n \quad \text{is Countable.}
\]

It is sufficient to show that \( A \) is countable.

since each \( A_n \) is Countable...

\[
i f_n: \mathbb{Z}^+ \rightarrow A_n \text{ which is surjective map,}
\]
and choose a surjection function \( g: \mathbb{Z}^+ \rightarrow \mathbb{N} \text{ (} \mathbb{Z}^+ \text{ is Countable)} \),

which is true for every \( n \).

Define \( h: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \cup A_n = A \) by the equation
\[
h(nm) = g_{cn}(m).
\]
Clearly \( h \) is surjective map.
Let \( a \in \mathbb{N} \) be any element, \( a \neq 0 \), for some \( j \),

\[
\rightarrow \exists f_j : z \rightarrow y \text{ if } f_j \text{ is surjective map, }
\]

\[
\rightarrow \exists \bar{m} \in \mathbb{Z} \mid \bar{m} + f_j(m) = a \quad \text{if } h(j,m) = f_j(y)(m) 
\]

Since \( \mathbb{N} \times \mathbb{N} \) is countably infinite and therefore \( U \mathbb{N} \) is countable,

it is also countably infinite.

and hence \( f_j \in U \mathbb{N} \) is countable.

Therefore, countable union of countable sets is countable.

E1. Show that \( \mathbb{Q} \) is countable.

Define a map \( h : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q} \) by \( h(x,y) = \frac{x}{y} \).

This map is well-defined:

For injective:

Suppose \( h(x_1,y_1) = h(x_2,y_2) \)

\[ \frac{x_1}{y_1} = \frac{x_2}{y_2} \]

\[ x_1y_2 = x_2y_1 \]

If \( x + a = y + b \), then the rational is not same. Similar for \( y + a = x + b \).

\[ h(x,y) = h(x',y') \]

\[ \Rightarrow A \text{ is injective map.} \]

By theorem (X) \( \Rightarrow h \) is surjective.

\[ \mathbb{Q} \text{ is countable.} \quad \text{and where } A \text{ is countably infinite.} \]

Similarly \( \mathbb{Q} \) is countable.

E2. Show that \( \mathbb{Q} \) is countable.

We know that \( \mathbb{Q}^+ \) and \( \mathbb{Q}^- \) are countable.

Also \( \mathbb{Q} \) is countable.

We know that countable union of countable sets is countable.

\[ \mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \] is countable.

\[ \Rightarrow \mathbb{Q} \text{ is countable.} \]

In a finite product of countable sets is countable.

Proof: Let \( A_1, A_2, \ldots, A_n \) are countable sets.

Claim: \( A_1 \times A_2 \times A_n \) is countable.

We prove this result by using mathematical induction:

Step 1. For \( n = 2 \)

Let \( A_1 \) and \( A_2 \) are countable.

To show that \( A_1 \times A_2 \) is countable.
If \( A_1 \) or \( A_2 \) is empty then we are through otherwise, choose a function \( f: A_2 \to A_1 \) and define the function \( h: A_1 \times \{ 0, 1 \} \to A \times A_2 \) by the eqn

\[
h(nm) = (f(n), g(n))
\]

is a surjective map.

So that \( A \times A_2 \) is countable (by step 0).

Result: Assume that the result is true for \( 2 \leq n \).

\( \Rightarrow n \times A \times A_2 \times \cdots \times A_2 \) is countable.

Step 1: To show that \( A \times A \times \cdots \times A \) is countable

Since \( A \times A \times \cdots \times A \) is countable (by step 2),

\( A \) is countable.

\( \Rightarrow (A \times A \times \cdots \times A) \times A \) is countable (by step 2).

\( \Rightarrow A \times A \times \cdots \times A \times A \) is countable.

Hence a finite product of countable sets is countable.

Proof: Let \( x \) denote the two element set \( \{0, 1\} \). Then the set \( x^n \) is uncountable. (\( x^n \) is an arbitrary infinite product of \( x \)).

The set \( x^n \) is uncountable, it is sufficient to show that \( g: 2^n \to A \) is not surjective.

To prove that \( g: 2^n \to A \) is not surjective.

Let us denote \( g(n) \) as follows:

\[
g(n) = (x_1, x_2, x_3, \ldots, x_n)
\]

where each \( x_i \) is either 0 or 1.

Then we define an element \( y = (y_1, y_2, \ldots, y_{n+1}) \) of \( x^{n+1} \) by letting

\[
y = \begin{cases} 0 & \text{if } x_n = 1 \\ 1 & \text{if } x_n = 0 \end{cases}
\]

Then \( y \) cannot have any pre-image in \( 2^n \).

This means there does not exist any surjective map from \( 2^n \) to \( x^n \).

Hence \( x^n \) is uncountable.

Let us write the numbers \( x_i \) in a rectangular array. The particular elements \( x_{ni} \) appear as the diagonal entries in this array. We choose \( y_i \) such that its \( i \)-th coordinate differs from the \( i \)-th diagonal entry \( x_{ni} \).
Ex. 5. \( Q \) is countably infinite.

\( \mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\} \) is a set of rationals.

Let \( A_n = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\} \) where \( n \in \mathbb{Z}^+ \).

So that \( A_n \) is a set of all those rationals whose denominator is \( n \).

Then there is a bijection from \( \mathbb{Z}^+ \) to \( A_n \) defined as

\[
f : \mathbb{Z}^+ \rightarrow A_n\]

where

\[
f(m) = \begin{cases} 
\frac{m}{2}, & \text{if } m \text{ is even,} \\
\frac{m-1}{2}, & \text{if } m \text{ is odd.}
\end{cases}
\]

This \( A_n \) is countable.

We know that a countable union of countable sets is countable.

Hence \( Q \) is countably infinite.

Thm. Let \( A \) be a set. There is no injective map \( f : \mathbb{Z}^+ \rightarrow A \).

and there is no surjective map \( g : A \rightarrow \mathbb{Z}^+ \) (where \( \mathbb{P}(A) \) is the set of all subsets of \( A \)).

Proof: Suppose that \( g : A \rightarrow \mathbb{P}(A) \) is surjective.

For any \( a, b \in A \), let \( g(a) \) and \( g(b) \) be arbitrary.

Define \( B = \{ a \in A \mid a \notin g(a) \} \).

Then \( B \) is a subset of \( A \) which is may be empty or may not be empty.

By construction, \( B \in \mathbb{P}(A) \).

\[
\Rightarrow \exists a \in A \text{ such that } g(a) = B.
\]

Then clearly \( a \in B \iff a \notin g(a) \).

\[
g(a) \supseteq B.
\]

Hence \( V. B \in \mathbb{P}(A) \) if \( a \in B \iff g(a) = B \).

Which is not possible. Hence \( g \) is not surjective.

Similarly \( f : \mathbb{P}(A) \rightarrow A \) is not injective.
A set $B$ is called to be well-ordered set with order relation $<$ if every non-empty subset of $B$ has smallest or least element.

Eq. $a < b$, $a \in B$ then $B$ is a well-ordered set.

1. Let $A$ be a set. The following statements about $A$ are equivalent.

2. There exists an injective function $f: \mathbb{Z}^+ \to A$

3. There exists a bijection of $A$ with a proper subset of itself.

4. $A$ is infinite.

Proof: $1 \implies 2$.

Let $f: \mathbb{Z}^+ \to A$ be an injective map.

Let the image set $f(\mathbb{Z}^+)$ be denoted by $B$, and let $f_m$ be denoted by $a_m$.

Because $f$ is injective,

$a_m = a_{m+1}$ if $m < m$.

Define $g: a \mapsto A - \{a_m\}$ by

$g(a_m) = a_{m+1}$, for $a_m \in B$

$g(x) = x$, for $x \in A - B$.

Then clearly $g$ is bijective map, hence $A = B$.

$A$ is a bijective map from $A$ to its proper subset.

$2 \implies 3$.

Let $g$ be a bijective map from $A$ to its proper subset then $A$ must be infinite set.

(Contrapositive of Corollary 6.3) If $A$ is finite, there is no bijection of $A$ with a proper subset of itself.

Let $A$ be an infinite set.

Let $a \in A$ be any element.

Define $f: \mathbb{Z}^+ \to \mathbb{A}$ such that $f_m \neq a$. 

And by induction.
Define $f(1), f(2), f(3), \ldots, f(n)$. 

Since $A$ is an infinite set, for some $n \in \mathbb{N}$, 

$A = \{ f(1), f(2), \ldots, f(n) \}$ is also an empty infinite subset. 

So, select any $a \in A - \{ f(1), f(2), \ldots, f(n) \}$. Set $f(n) = a$. 

Hence by induction, the map $f: \mathbb{N} \rightarrow A$ is injective.
Unit 2: Topological Spaces & Continuous Functions

A topological space on a set $X$ is a collection $\gamma$ of subsets of $X$ having the following properties:

i). $\emptyset$ and $X$ are in $\gamma$.

ii). The union of the elements of any sub-collection of $\gamma$ is in $\gamma$.

iii). The intersection of the elements of any finite sub-collection of $\gamma$ is in $\gamma$.

A set $X$ together with the topology $\gamma$ is called a topological space.

We denote it by $(X, \gamma)$.

It $X$ is a topological space with topology $\gamma$, we say that a subset $U$ of $X$ is an open set of $X$ belonging to the collection $\gamma$.

Using this terminology, one can say that a topological space is a set $X$ together with a collection of subsets of $X$, called open sets, such that $\emptyset$ and $X$ are both open and such that arbitrary unions and finite intersections of open sets are open.

Examples:

1. Let $X = \{a, b, c\}$

   $\gamma_1 = \{\emptyset, \{a, b\}\}$ is not a topology on $X$.

   $\gamma_2 = \{\emptyset, X, \{a\}\}$ is a topology on $X$.

   $\gamma_3 = \{\emptyset, X, \{a, b\}\}$ is a topology on $X$.

   $\gamma_4 = \{\emptyset, X, \{a, b, c\}\}$ is a topology on $X$.

   Indiscrete topology:

2. Let $X$ be a set then $\gamma = \{\emptyset, X\}$ is topology on $X$ called indiscrete or trivial topology on $X$.

3. Let $X = \{a, b, c\}$

   $\gamma_1 = \{\emptyset, X, \{a, b, c\}\}$ is topology on $X$.

   $\gamma_2 = \{\emptyset, X, \{a, b, c\}\}$ is topology on $X$. 

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\[ T_2 = \{ \emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\} \} \text{ is not topology on } X. \]

Because \( \{ \{a,b\}, \{b,c\} \} \) is a finite subcollection of \( T_2 \) but \( \{a,b\} \cap \{b,c\} \neq \emptyset \) \( \emptyset \neq \emptyset \).

\[ T_2 \text{ is not topology on } X. \]

\[ T_3 = \{ \emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\} \} \text{ is not topology on } X. \]

Because \( \{ \{a,b\}, \{b,c\} \} \) is a finite subcollection of \( T_3 \) but \( \{a,b\} \cap \{b,c\} = \{b\} \neq \emptyset \).

\[ T_3 \text{ is not topology on } X. \]

Ex: Let \( X \) be a set. Let \( T_f \) be the collection of all subsets \( U \) of \( X \) such that \( X - U \) either is finite or is all of \( X \).

\[ T_f = \{ U | U \subseteq X, X \setminus U \text{ finite or } X \setminus U = X \} \]

Then \( T_f \) is topology on \( X \) and is called finite complement topology.

Both \( X \) and \( \emptyset \) are in \( T_f \).

Since \( X - X \) is finite \( \Rightarrow X \in T_f \)

and \( X - \emptyset = X \Rightarrow \emptyset \in T_f \).

Let \( \{ U_i \}_{i \in I} \) be indexed family of non-empty elements of \( T_f \).

Claim: Claim: \( \bigcup_{i \in I} U_i \in T_f \).

Now \( X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X \setminus U_i) = \text{ finite} \)

because each \( X - U_i \) is finite

\( \Rightarrow U_i \in T_f \).

Let \( U_1, U_2, \ldots, U_n \) be non-empty members of \( T_f \).

Claim: \( \bigcap_{i=1}^{n} U_i \in T_f \).
Now
\[ x - \bigcap U = \bigcup (x - U_i) \quad (a) \]
\[ \text{Each } x - U_i \text{ is finite then R.H.S of eqn (a) is finite} \]
\[ \Rightarrow \bigcap U_i \in \tau_f \]

**Comparison of Topology:**

Let \( \tau \) and \( \tau' \) be two topologies on a given set \( X \). If \( \tau \subseteq \tau' \), then we say that \( \tau' \) is a finer (coarser or stronger) than \( \tau \).

If \( \tau \) is properly contained in \( \tau' \) (i.e., \( \tau \subset \tau' \)), then \( \tau' \) is said to be strictly finer than \( \tau \).

Alternatively, if \( \tau \subseteq \tau' \), we say that \( \tau \) is coarser (or smaller or weaker) than \( \tau' \).

**Example:**

Let \( X = \{a, b\} \)

\( \tau = \{\emptyset, X, \{a\}, \{b\}\} \quad \tau' = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \)

both \( \tau \) and \( \tau' \) topologies on \( X \).

But \( \tau \not\subseteq \tau' \) and \( \tau' \not\subseteq \tau \)

\( \Rightarrow \tau \) and \( \tau' \) are not comparable.

**Example:**

Let \( X = \{a, b\} \)

\( T_e = \{ \emptyset, \{a\}, \{b\}, X \} \quad 	ext{trivial topology or discrete topology on } X \)

\( T_d = \{ \emptyset, \{a\}, \{b\}, X \} \quad \text{discrete topology on } X \).

Then \( T_e \) is finer than \( T_d \).

\( T_e \) and \( T_d \) are finer than \( T_f \).

\( T_d \) is finer than \( \tau \) and \( \tau' \).

**Ex:**
Let \( X \) be a set and \( T_e = \{ U \subseteq X | X \cup U \text{ is countable or } X \cup U = X \} \).

Then \( T_e \) is topology on \( X \).

**Con:**
Let \( X \) be a set.

\( T_e = \{ U \subseteq X | X \cup U \text{ is countable or } X \cup U = X \} \).

1. If \( U \subseteq X \Rightarrow X \cup U = X \) which is countable.

   \( \Rightarrow X \in T_e \).
Let \( U = \emptyset \) \( \Rightarrow \) \( x \in U = x \in \mathcal{C} \).

\[ \emptyset, x \in \mathcal{C} \]

2. Let \( \{ U_x \} \) be the collection of \( \mathcal{C} \).

Let \( U = U \cup \mathcal{U} \).

Case 1:

Now \( X - U = \bigcap \{ x : x \in U \} \).

We know that arbitrary intersection of countable set is countable.

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**Basis of a Topology:**

Let \( x \) be a set. A basis for a topology on \( x \) is a collection of "subsets" of \( x \) (called basis elements) such that:

1) For every \( x \in X \) there is some basis element containing \( x \).
2) If \( x \in B \cap B' \), and \( B, B' \) are in \( B \), then there is a basis element \( B \) such that \( x \in B \cap B' \).

**Examples:**

1) Let \( X = \{ 0, 1, 2 \} \).

2) \( B = \{ 0, 3, 4, 5 \} \). Then \( B \) is basis for topology on \( X \).

Condition 1) holds.

There is no element of \( X \) contained in the intersection of two basis elements so condition 2) holds.

3) \( B = \{ 1, 3, 4, 5 \} \).

Condition 1) holds.

\( 0, 5 \in B, 0 \notin B \).

Condition 2) fails.

It is not basis for topology on \( X \).
Let $X$ be the set of real numbers. If $B = \{ (a, b) \mid a < b \}$, then $B$ is a basis for the topology on $X$.

Let $x \in (a, b)$, then $x \in (x - 1, x + 1) \in B$ for $x = (a + b)/2$.

Case 1:
- $a < c < d < b$
  
Case 2:
- $a < c < b < d$

Note: If $B$ satisfies these two conditions, then we define the topology $\mathcal{T}$ generated by $B$ as follows: A subset $U$ of $X$ is said to be open in $X$ (that is, to be an element of $\mathcal{T}$) if, for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

Let $\mathbb{C}$ be the collection of all circular regions (circular regions) in the plane. Then $\mathbb{C}$ satisfies both conditions for a basis. The second condition is illustrated below. Any open set in the topology generated by $\mathbb{C}$, a subset $U$ of the plane, is open if every $x$ in $U$ lies in some circular region contained in $U$. 
e. Let $B'$ be the collection of all rectangular regions (interior of rectangles) in the plane, where the rectangles have sides parallel to the coordinate axes. Then $B'$ satisfies both conditions for a basis.

The second condition in the text below Fig. 1 is this case, the condition is trivial because the intersection of any two basis elements is itself a basis element (as empty). The basis $B'$ generates the same topology on the plane as the basis $B$ given in the previous example.

* Topology $T$ generated by basis $B$.
Let $(X, Y)$ be any topology and $S, D$ a basis for a topological space $X$. Then the topology $T := \mathcal{B}$ is said to be generated by basis $B$ if for every $x \in U, C \in \mathcal{B}$ there is a $\mathcal{B}_x$ such that $x \in \mathcal{B}_x \subseteq U$.

Remark: Let $X$ be any topological space with topology $\mathcal{T}$ and $B$ is a basis for a topology $\mathcal{T}$ on $X$, then $\mathcal{B}$.

Remark: If $B'$ is a basis which generates a topology $\mathcal{T}$, then every basis element is itself an element of $\mathcal{T}$. Hence, every element of a basis $B'$ also becomes an open set.
Define $T = \{ U \subseteq X \mid \forall x \in U \exists \mathcal{B} \in \mathcal{B} \text{ such that } x \in \mathcal{B} \subseteq U \}$.

In topology on $X$, $T$ is called as topology generated by basis $\mathcal{B}$.

For every $x \in X$, by condition of definition of basis of $\mathcal{B}$, there exists $\mathcal{B} \in \mathcal{B}$ such that $x \in \mathcal{B} \subseteq U$.

(i) Let $U \neq X$.

Let $U = U_1 \times U_2$.

Want: $U, U_1, U_2$.

Let $x \in U_1 \times U_2$.

Then $U \subseteq U_1 \times U_2$ for some $x$.

If $x \in U_1 \times U_2$, let $x \in U_1 \times U_2$.

(ii) Let $U_1, U_2$.

Want: $U_1, U_2 \subseteq U$.

Let $x \in U_1 \times U_2$.

Then $U \subseteq U_1 \times U_2$ and $x \in U_1 \times U_2$ but $U_1, U_2 \subseteq U$.

$\Rightarrow \exists \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}$ such that $x \in \mathcal{B}_1 \subseteq U$ and $x \in \mathcal{B}_2 \subseteq U_2$.

By $x \in \mathcal{B}_1 \subseteq U$ and $x \in \mathcal{B}_2 \subseteq U_2$, $x \in \mathcal{B}$.

Let $V_1 = \mathcal{B}_1, V_2 = \mathcal{B}_2$.

Assume that $U_1 \cup U_2 \neq V_1 \cup V_2$.

Want: $U_1 

Let $U_1 \cup U_2 = V_1 \cup V_2$.

Let $U_1 \cup U_2 = V_1 \cup V_2$.

$\Rightarrow \mathcal{B} \subseteq \mathcal{B}$.

Let $x \in \mathcal{B}$.

$\Rightarrow (U_1 \cup U_2) \cap \mathcal{B} \neq U_1 \cup U_2$.

$= T$ is topology on $X$, and $T$ is called the topology generated by basis $\mathcal{B}$, where $\mathcal{B}$ is basis called basis for $T$.

And every member of $\mathcal{B}$ is called a basis element.

(iii) Let $X$ be discrete topological space.

$\exists \mathcal{B} = \{ \{ x \} \mid x \in X \}$; then $\mathcal{B}$ is a basis for discrete topological space.
Lemma: Let \( X \) be a set; let \( \mathcal{B} \) be a basis for a topology \( \tau \) on \( X \). Then \( \tau \) equals the collection of all unions of elements (or members) of \( \mathcal{B} \).

Proof: We know that \( \tau = \{ \bigcup \mathcal{B} | \mathcal{B} \in \mathcal{B} \} \) for \( x \in \bigcup \mathcal{B} \) such that \( x \in \bigcup \mathcal{B} \).

Let \((x, \mathcal{T})\) be any topological space, and \( \mathcal{B} \) be a basis for topology \( \mathcal{T} \) for any \( U \in \mathcal{T} \) such that \( x \in \bigcup \mathcal{B} \) with \( x \in \bigcup \mathcal{B} \).

We know every element of \( \mathcal{T} \) is an open set.

From eqn \( 1 \) \( \bigcup \mathcal{B} \in \mathcal{T} \) and \( x \in \bigcup \mathcal{B} \),

Conversely, also for \( U \in \mathcal{T} \) and \( x \in \bigcup \mathcal{B} \) such that \( x \in \bigcup \mathcal{B} \).

From eqn \( 1 \), \( 2 \), and \( 3 \), we get \( x \in \bigcup \mathcal{B} \).

Let \( \mathcal{E} \) be a basis for the topology of \( X \).

Proof: Let \( x \in X \)

\( X \) itself is open in \( X \).

By definition of \( \mathcal{E} \), if \( C \in \mathcal{E} \), such that \( x \in C \) in \( X \).

Thus \( C \in \mathcal{E} \), such that \( x \in C \) in \( X \).

Let \( C_1, C_2 \in \mathcal{E} \) and \( x \in C_1 \cap C_2 \).

Since \( C_1, C_2 \in \mathcal{E} \),

\( C_1 \cap C_2 \in \mathcal{E} \).

\( C_1 \cap C_2 \) is open set.

By definition of \( \mathcal{E} \) if \( C_1, C_2 \in \mathcal{E} \), such that \( x \in C_1 \cap C_2 \).

\( \mathcal{E} \) is a basis.

Let \( \mathcal{Y} \) be the topology generated by basis \( \mathcal{E} \).

We know that \( \mathcal{Y} = \{ \bigcup \mathcal{E} | \mathcal{E} \in \mathcal{E} \} \) for \( x \in \bigcup \mathcal{E} \) such that \( x \in \bigcup \mathcal{E} \).

Let \( U, Y \in \mathcal{Y} \).
Let \( x \in U \)

By definition of \( \mathcal{E} \), \( x \in \mathcal{E} \), such that \( x \in \mathcal{E} \cap U \)

\[ x \in U \cap \mathcal{E} \]

5.

Conveniently suppose that \( U \neq \emptyset \).

Since \( \mathcal{T} \) is topology generated by \( \mathcal{E} \) basis \( \mathcal{E} \),

\[ U \cup U \mathcal{E} \text{ where } U \mathcal{E} \]

But \( E \) is a subcollection of \( \mathcal{E} \) ie \( \mathcal{E} \cap \mathcal{T} \).

\[ E \in \mathcal{T}, \forall x \in E \]

\[ U \cup U \mathcal{E} \subseteq \mathcal{T} \text{ (i.e. topology} \)

\[ U \subseteq \mathcal{E} \]

6.

From eqn 0 @ we get

\[ \mathcal{E} = \mathcal{T} \]

Lemma - Let \( \mathcal{B} \) and \( \mathcal{B}' \) be bases for the topology \( \mathcal{T} \) and \( \mathcal{T}' \) respectively, on \( X \).

1. \( \mathcal{T}' \) is finer than \( \mathcal{T} \).

2. For each \( x \in X \) and each basic element \( B \in \mathcal{B} \) containing \( x \), there is a basis element \( B' \in \mathcal{B}' \) such that \( x \in B' \subseteq B \).

Proof:

1) \( \Rightarrow \) 2)

Suppose that \( \mathcal{T}' \) is finer than \( \mathcal{T}' \), i.e. \( \mathcal{T}' \subseteq \mathcal{T} \).

Let \( x \in X \) and \( B \in \mathcal{B} \) such that \( x \in B \).

but \( B \in \mathcal{B}' \) \( \Rightarrow \) \( B \in \mathcal{B}' \subseteq \mathcal{B} \)

Since \( \mathcal{T}' \) is topology generated by \( \mathcal{B}' \), \( \exists B' \in \mathcal{B}' \) such that \( x \in B' \subseteq B \).

3) \( \Rightarrow \) 2)

Suppose that 2) holds.

Claim. \( \mathcal{T}' \subseteq \mathcal{T} \).

Let \( U \in \mathcal{T} \).

Let \( x \in U \) \( \exists B \in \mathcal{B} \) \( B \subseteq \mathcal{B} \) such that \( x \in B \subseteq U \)

\[ \left( \because B \text{ is basis for topology } \mathcal{T} \text{ on } X \right) \]

By 3), \( \exists B' \subseteq B \) such that \( x \in B' \subseteq U \)

\[ \Rightarrow x \in B' \subseteq B \]

\[ \Rightarrow U \subseteq T \]

\[ \Rightarrow T \subseteq \mathcal{T} \]

ie \( \mathcal{T}' \) is finer than \( \mathcal{T} \).
Defining: Let $B$ be the collection of all open intervals in the real line, $(a,b) = \{ x \in \mathbb{R} \mid a < x < b \}$, the topology generated by $B$.

$B$ is called the standard topology on the real line $\mathbb{R}$.

1. If $B^\prime$ is the collection of all half-open intervals of the form $(a,b] = \{ x \in \mathbb{R} \mid a < x \leq b \}$, the topology generated by $B^\prime$ is called the lower-limit topology on $\mathbb{R}$. We denote it by $\tau_L$.

Let $K = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$ and let $B^K$ be the collection of all open intervals, $(a,b)$, as well as intervals of the form $(a,b)-K$. The topology generated by $B^K$ is called the $K$-topology on $\mathbb{R}$. We denote it by $\tau_K$.

Lemma: The topologies of $\tau_K$ and $\tau_L$ are strictly finer than the standard topology on $\mathbb{R}$.

Proof: Let $\tau$, $\tau_L$, and $\tau_K$ be the topologies of $\mathbb{R}$, $\tau_L$, and $\tau_K$.

Want: $\tau < \tau_K$ and $\tau < \tau_L$.

Let $(a,b)$ be a basis element for $\tau$. Let $x \in (a,b)$ then $x \in \tau_K$.

Further $(x, (a,b)] \subset (a,b)$.

By above lemma,

$\tau \subset \tau_K$.

Consider a basis element $(x,b)$ of $\tau_K$ containing $x$.

If $(c,d)$ is any open interval containing $x$ then

$(c,d) \notin \tau_K$.

$\Rightarrow$ $\tau < \tau_K$.

$\tau \subset \tau_L$.

$\Rightarrow$ $\tau \neq \tau_K$.

Let $(a,b)$ be a basis element for $\tau_L$.

Claim: $\tau \neq \tau_K$.

Let $x \in (a,b)$ then

But $(a,b) \subset \tau_K$ also $x \in (a,b)$.

$\Rightarrow x \in \tau_K$.

Let $B = (a,b)-K$.

$\Rightarrow B \subset \tau_K$.

Any open interval $(a,b)$ containing $x$ contains $\frac{1}{m}$ for

infinitely many $m$.

$\Rightarrow (a,b) \cap B = (a,b)-K$

$\Rightarrow \tau < \tau_K$.

$\Rightarrow \tau < \tau_L$.

$\tau_K$ is strictly finer than $\tau_L$. 

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Example 23. Lower limit topology and \( k \)-topology on \( \mathbb{R} \) are not comparable.

Solution: Let \( \mathcal{T}_k \) be lower limit topology and \( \mathcal{T}_L \) be topology on \( \mathbb{R} \) respectively.

Let \( O = (-1, 1) - k \)

\( O \) is a basis element of \( \mathcal{T}_k \)

Further \( O \in \mathcal{B} \)

Any basis element \( (a, b) \) of \( \mathcal{T}_L \) containing \( O \) contains \( O \) for many \( n \)'s. Hence \( (a, b) \in (1, 1) - k \).

\( \mathcal{T}_k \subseteq \mathcal{T}_L \).

Conversely \( (0, 1) \) is a basis of \( \mathcal{T}_L \) containing \( O \).

Any basis element of \( \mathcal{T}_L \) containing \( O \) contains some negative numbers.

\( \Rightarrow \) such basis element is not a subset of \( (0, 1) \).

\( \Rightarrow \mathcal{T}_k \not\subseteq \mathcal{T}_L \).

Thus \( \mathcal{T}_k \) and \( \mathcal{T}_L \) are not comparable.

**Example 24.** Arbitrary intersection of topologies on \( X \) is a topology.

Solution: Let \( \gamma \) be a collection of topologies on \( X \).

Let \( \gamma = \bigcup_{\alpha \in J} \mathcal{T}_\alpha \)

Claim: \( \gamma \) is topology on \( X \).

Since \( \phi \times X \in \mathcal{T}_{\alpha} \) and \( X \times \phi \in \mathcal{T}_{\alpha} \)

\( \Rightarrow \phi \times X \in \gamma \) and \( X \times \phi \in \gamma \)

Let \( U \bigcup_{\alpha \in J} V_{\alpha} \in \gamma \), \( V_{\alpha} \subseteq \mathcal{T}_{\alpha} \)

\( \Rightarrow U \bigcup_{\alpha \in J} V_{\alpha} \subseteq \gamma \)

Since each \( \mathcal{T}_{\alpha} \) is topology on \( X \)

\( \Rightarrow U \bigcup_{\alpha \in J} V_{\alpha} \subseteq \gamma \)

\( \Rightarrow \gamma \) is topology on \( X \).
Ex. Show that if \( A \) is a basis for a topology on \( X \), then the topology generated by \( A \) equals the intersection of all topologies on \( X \) that contain \( A \). Prove the same if \( A \) is a subbasis.

Solution: Let \( \tau \) be a topology generated by \( A \). Then \( A \subseteq \tau \).

Let \( \tau' \) be the intersection of all topologies containing \( A \).

- \( \tau' \) is contained in all these topologies.
- \( \tau \subseteq \tau' \).

Conversely, suppose that \( U \in \tau' \).

- \( U = \bigcup_{A \in A} A \) (Lemma 8)

But \( A \in \tau \) implies that \( A \) is topology containing \( A \).

- \( U = \bigcup_{A \in A} A \) is a topology containing \( A \).

Thus, \( U \in \tau \).

From (1) and (2), we get \( \tau = \tau' \).

Ex. Show that \( B = \{ (a, b) \mid a < b \} \), \( a \) and \( b \) rational, is a basis that generated the standard topology on \( \mathbb{R} \).

Solution: Let \( x \in \mathbb{R} \).

- Let \( \eta_{1}, \eta_{2} \) be two rationals such that \( \eta_{1} < x < \eta_{2} \).

Then \( x \in (\eta_{1}, \eta_{2}) \) and \( (\eta_{1}, \eta_{2}) \in B \).

Let \( (a_{1}, b_{1}) \) and \( (a_{2}, b_{2}) \) be such that \( x \in (a_{1}, b_{1}) \cap (a_{2}, b_{2}) \).

- \( x \in (a_{1}, b_{1}) \) and \( x \in (a_{2}, b_{2}) \).

Let \( a_{0} = \max \{ a_{1}, b_{1} \} \) and \( b_{0} = \min \{ b_{1}, b_{2} \} \).

- \( a_{0} \) and \( b_{0} \) are rationals.

Further, \( x \in (a_{0}, b_{0}) \).

- \( \mathbb{R} \) is a basis.

Let \( \tau' \) be the topology generated by \( \mathbb{R} \).

Let \( \tau \) be the standard topology on \( \mathbb{R} \).

Then \( \tau = \tau' \).
Every element of $G$ is in $\gamma$.
\[ \gamma = \gamma' \cap \gamma'' \]

Consequently let $x \in \mathbb{R}$

Let $c(a,b) \in \gamma$ such that $x \in c(a,b)$.
\[ a \leq x \leq b \]

Let $r, s$ be reals such that $a < r < s < b$.
\[ x \in c(a, s) \cap c(s, b) \]
\[ r \leq x \leq s \]

From eqn (1) and (2) we get
\[ r \leq x \leq s \]

**Subbasis:**

Let $G$ be collection of subsets of $X$ whose union equals $X$.

Let $G$ be collection of finite intersections of elements of $S$.

Any $G$ is a basis for $X$.

Let $x \in X$. We have $X = \bigcup_{G \in G} S_G$

$\Rightarrow x \in S$ for some $S \in G$

Let $S \in G$.

$\Rightarrow x \in S$ and $S \in G$.

Let $B_1 = S_{b_1} - S_{b_2} \in G$

Let $B_2 = S_{b_2} - S_{b_1} \in G$.

Let $x \in B_1 \cap B_2$.

$\Rightarrow x \in (S_{b_1} - S_{b_2}) \cap (S_{b_2} - S_{b_1}) = S_{b_2} - S_{b_2} \in G$.

Let $B_1 \in G$ and $x \in B_1 \in G$.

$\Rightarrow x \in B_1 \in G$.

$\gamma$ be the topology generated by $G$.

$\Rightarrow$ $\gamma$ is the collection of unions of finite intersections of elements of $S$.

We say that $S$ is a subbasis for $\gamma$.

**The order topology:**

Suppose $X$ be a set with simple order $<$. Let $a \leq X$ such that $a \in b$ then
\[ (a,b) = \{ x \in X \mid a \leq x \leq b \} \]

\[ [a,b) = \{ x \in X \mid a < x \leq b \} \]
\[(a, b) = \{ x \in \mathbb{R} : a < x < b \} \quad [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}\]

These sets are called intervals.

- \((a, b)\) - open interval
- \([a, b]\) - closed interval
- \([a, b), (a, b]\) - half-open intervals, or half-closed intervals.

**Definition:** Let \(X\) be a set with a simple order relation, assume \(X\) has more than one element. Let \(\mathcal{B}\) be the collection of all sets of the following types:

1. All open intervals \((a, b)\) in \(X\).
2. All intervals of the form \([a, b)\), where \(a\) is the smallest element of \(X\) (if any) of \(X\).
3. All intervals of the form \([a, b]\), where \(b\) is the largest element of \(X\) (if any) of \(X\).

The collection \(\mathcal{B}\) is a basis for a topology on \(X\), which is called the ordered topology.

**Example:** \(\mathbb{R}\) is an ordered set.

Since \(\mathbb{R}\) does not have largest, smallest element, the basis for order topology is the collection of the intervals \((a, b)\).

**Order topology** = standard topology on \(\mathbb{R}\).

**Consider** the set \(\mathbb{R} \times \mathbb{R}\) in the dictionary order, we shall denote the general element of \(\mathbb{R} \times \mathbb{R}\) by \((x, y)\).

The set \(\mathbb{R} \times \mathbb{R}\) has neither a largest nor a smallest element. The ordered topology on \(\mathbb{R} \times \mathbb{R}\) has a basis the collection of all open intervals of the form \((a, b) \times (c, d)\) for \(a < c\) and \(a < c\) and \(b < d\). These two types of intervals are indicated in fig c.
Definition: If \( X \) is an ordered set, and \( a \) is an element of \( X \), there are four subsets of \( X \) that are called the rays determined by \( a \). They are the following:

\[
\begin{align*}
(a, a) &= \{ x \mid x < a \} \\
(-a, a) &= \{ x \mid x < a \} \\
[a, a] &= \{ x \mid x > a \} \\
(-a, a] &= \{ x \mid x \leq a \}
\end{align*}
\]

Sets of the first two types are called open rays, and sets of the last two types are called closed rays.

The Product Topology on \( X \times Y \).

Definition: Let \( X \) and \( Y \) be topological spaces. The product topology on \( X \times Y \) is the topology having as basis the collection \( \mathcal{B} \) of all sets of the form \( U \times V \), where \( U \) is an open subset of \( X \) and \( V \) is an open subset of \( Y \).

That is, \( \mathcal{B} = \{ U \times V \mid U \text{ open in } X \text{ and } V \text{ open in } Y \} \) is basis for \( X \times Y \), product topology \( X \times Y \), but \( \mathcal{B} \) is not topology on \( X \times Y \).

Counter example:

The union of the two rectangles is not a product of two sets, so it cannot belong to \( \mathcal{B} \), however it is open in \( X \times Y \).

Lemma: If \( \mathcal{B} \) is a basis for the topology of \( X \) and \( \mathcal{C} \) is a basis for the topology of \( Y \), then the collection

\[
\mathcal{D} = \{ B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C} \}
\]

is a basis for the topology of \( X \times Y \).
Proof: Let $\mathcal{Y}$ be the product topology, then it is generated by the collection $\mathcal{G} = \{ U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y \}$.

Since every member of $\mathcal{B}$ is open in $X$ and every member of $\mathcal{C}$ is open in $Y$,

$\Rightarrow$ every member of $\mathcal{B} \times \mathcal{C}$ of $\mathcal{B} \times \mathcal{D}$ is a member of $\mathcal{G}$ and is a member of $\mathcal{Y}$.

$\Rightarrow \mathcal{D} \subseteq \mathcal{Y}$.

Let $W \subseteq \mathcal{Y}$ and $x, y \in W$.

$\exists U \text{ open in } X \text{ and } V \text{ open in } Y$ such that $x \in U \times V$.

We have $x \in U \times V$ as $U$ and $V$ are open in $X$ and $Y$ respectively and $\mathcal{B}$ and $\mathcal{C}$ are bases for $X$ and $Y$ respectively.

$\exists E, B \in \mathcal{B}$ and $C, E \in \mathcal{C}$ such that $x \in E \times B$ and $y \in C \times V$.

$\Rightarrow x \in E \times B$ and $y \in C \times V$.

$\Rightarrow E \times B \subseteq \mathcal{D}$.

$\Rightarrow E \times B$ is a basis for the topology $\mathcal{Y}$ of $X \times Y$.

* Projection Maps:*

Let $\pi_1 : X \times Y \longrightarrow X$ be defined by the equation $\pi_1(x, y) = x$.

Let $\pi_2 : X \times Y \longrightarrow Y$ be defined by the equation $\pi_2(x, y) = y$.

The maps $\pi_1$ and $\pi_2$ are called the projection of $X \times Y$ onto its first and second factors, respectively.

$\Rightarrow \forall U \text{ open in } X, f(U)$ is open in $Y$.

Ex. Prove that $\pi_1, \pi_2$ are open maps.

*Proof:*

We have $\pi_1 : X \times Y \longrightarrow Y$ be defined by $\pi_1(x, y) = x$.

Let $W$ be open set in $X \times Y$.

$\Rightarrow W = \bigcup \{ U \times V : U \text{ open in } X \text{ and } V \text{ open in } Y \}$.

$\Rightarrow \pi_1(W) = \bigcup \{ \pi_1(U \times V) : U \text{ open in } X \text{ and } V \text{ open in } Y \}$.

$\Rightarrow \pi_1(W) = \bigcup \{ U \times V \text{ open in } X \}$.

$\Rightarrow \pi_1$ is an open map.
Similarly, $T_2$ is an open map.

**Def:** $(X,d)$ is a metric space.

$$d(x,y) = \begin{cases} \epsilon & \text{if } x \in U \text{ and } d(x,y) < \epsilon \\ \text{u} & \text{if } x \in U 
$$

$U$ is open in $X$ if every $x \in U$ is a ball $B(x_{\epsilon}) \in U$.

**Properties of open set:**

1. If $x$ and $y$ both are open in $X$.
2. $\{A_x\}$ is a collection of open sets on $X$.

Then $\bigcup_{x \in A} x$ is open. i.e. the arbitrary union of open sets is open.

**Finite intersection of open sets is open:**

**Theorem:** Let $X, Y$ be topological spaces. Let $\mathcal{T}_X(U)$ be open in $X$, $\mathcal{T}_Y(V)$ be open in $Y$. $\mathcal{T}_X(U) \cap \mathcal{T}_Y(V)$ is a subbasis for the product topology on $X \times Y$.

**Proof:** Let $T'$ be the topology on $X \times Y$ whose subbasis is $\mathcal{T}_X(U) \cap \mathcal{T}_Y(V)$.

Let $T$ be the topology on $X \times Y$ whose subbasis is $\mathcal{T}_X(U) \cup \mathcal{T}_Y(V)$.

Let $W \in T$. Then $W = T_x(U) \cup T_y(V)$.

Let $W \in T'$. Then $W = T_x(U) \cap T_y(V)$.

Suppose $W = T_x(U) \times T_y(V)$.

$W$ is open in $X \times Y$ as $T'$ is open in $Y$.

$T' \supseteq T$.

**Proof:**

$T' \subseteq T$.

Let $W \in T'$. Then $W = T_x(U) \cap T_y(V)$.

Let $H = U \times V$ where $U$ is open in $X$ and $V$ is open in $Y$.

$H \in T'$. Then $H = T_x(U) \cap T_y(V)$.

$T_x(U) \cap T_y(V) \subseteq T_x(U) \cup T_y(V)$.

$T_x(U) \cup T_y(V) \subseteq T_x(U) \cap T_y(V)$.

Thus, a finite intersection of element of $T'$.

$H \subseteq T'$. From $\mathcal{T}_X(U) \cup \mathcal{T}_Y(V)$ we get $T' \subseteq T$. Thus $T' \subseteq T$. 

$T' \subseteq T$.
The Subspace Topology:

Definition: Let \( X \) be a topological space with topology \( Y \). If \( Y \) is a subset of \( X \), the collection \( Y_\gamma = \{ \gamma \cup V \mid V \in Y \} \) is a topology on \( Y \), called the 
Subspace topology. With this topology \( Y \) is called a subspace of \( X \). 
Its open sets consist of all intersections of open sets of \( X \) with \( Y \).

Check that \( Y_\gamma \) is a topology on \( Y \).

i) \( \emptyset \cup Y = \emptyset \in Y_\gamma \) \( \quad \left( \because \emptyset \text{ is open in } X \right) \)

ii) \( X \cup Y = X \in Y_\gamma \) \( \quad \left( \because X \text{ is open in } X \right) \)

iii) Let \( \{Y_\gamma\} \) be a collection of members of \( Y_\gamma \) 

For every \( i \), \( A_i = U \cup Y \) where \( U_i \) is open in \( X \)

\[ U_X = \bigcup (U \cup Y) = (U \cup Y) \gamma \in Y_\gamma \] \( \quad \left( \because U \gamma \text{ is open in } X \right) \)

iv) Let \( A_1, A_2, \ldots, A_n \in Y_\gamma \)

\( \exists U_1, U_2, \ldots, U_n \) open in \( X \) such that

\[ A_i = U_i \cup Y \quad \text{for } i \in \{1, 2, \ldots, n\} \]

\[ \bigcap_{i=1}^{A_i} Y = \left( \bigcap_{i=1}^{U_i} Y \right) \gamma \in Y_\gamma \]

\( \left( \because U_i \text{ is open in } X \right) \)
Thus $\gamma$ is a topology on $\gamma$.

$\gamma$ is called subspace topology if $\gamma$ is called subspace of $X$.

6. Let $\gamma = \mathbb{R}$. Let $(a, b)$ be an open interval in $\mathbb{R}$.

$(a, b) \cap \gamma = \{ (a, b) \cap \gamma \} \cap \gamma = \{ (a, b) \cap \gamma \} \cap \gamma = \{ (a, b) \cap \gamma \}$

Thus $(a, b)$ is open in $\gamma$ but not open in $\mathbb{R}$.

Remark. Let $\gamma$ be a subset of $X$.

i) Let $a \in \gamma$ such that $a$ is open in $X$.

Then $a = \gamma$ is open in $\gamma$ ("$a$ is open in $X$"").

ii) $\gamma$ is open in $\gamma$, $\gamma$ open in $X$, then $\gamma$ is open in $X$.

Proof. Since $\gamma$ is open in $\gamma$,

$\gamma = \bigcup_{\gamma} \gamma$ for some $\gamma \in \gamma$ open in $X$.

As $\gamma, \gamma$ are open in $X$, $\gamma$ open in $X$.

$\gamma = \bigcup_{\gamma} \gamma$ is open in $X$.

Thus $\gamma$ is open in $\gamma$.

Theorem: If $B$ is a basis for the topology of $X$, then the collection $\gamma_B = \{ \gamma \}$ is a basis for the subspace topology on $\gamma$.

Proof. Let $\gamma$ be a topology on $X$.

Then $\gamma_B = \{ \gamma \}$ is a subspace topology on $\gamma$.

If $B \in B$, then $B \in \gamma_B$,

$\Rightarrow B \in \gamma_B \gamma_B$.

$\Rightarrow B \in \gamma_B \gamma_B$.

$\Rightarrow B \in \gamma_B \gamma_B$. 

\[ \]
Let \( H \subseteq Y \) and \( z \in W \).

- \( z = u'z \) for some \( u' \in Y \).

as \( H \subseteq Y \) is a basis for \( Y \), \( z \) is a basis element.

- Let \( B \subseteq U \) be such that \( x \in B \).

thus \( z \in B \) is a basis element.

Thus, \( B \subseteq U \) contains \( x \) which is a subset of \( W \).

- \( B \subseteq U \) is a basis for \( Y \).

in \( B \subseteq U \) is a basis for subspace topology on \( Y \).

**Theorem:** If \( A \) is a subspace of \( X \) and \( B \) is a subspace of \( Y \) then the product topology on \( A \times B \) is same as the topology \( A \times B \) inherits as a subspace of \( X \times Y \).

**Proof:** Let \( U \times V \) be a general basis element of \( X \times Y \).

- \( U \) is open in \( A \) and \( V \) is open in \( B \).

By definition of subspace topology,
\( U = U' \cap A \) for some \( U' \) open in \( X \).
and \( V = V' \cap B \) for some \( V' \) open in \( Y \).

- \( U \times V = (U' \times V') \cap (A \times B) \)

\( U' \times V' \) is a basis element of \( X \times Y \).

\( (U' \times V') \cap (A \times B) \) is a basis element of subspace topology.

Thus, basis elements of product topology on \( A \times B \) are basis elements for subspace topology.

Similarly, basis elements for subspace topology are basis elements for product topology.

- Product topology on \( A \times B = \) subspace topology on \( A \times B \).

**X - Ordered Set:** \( X \times X \). The \( X \) has the order topology. The subspace topology on \( X \) may not be equal to the order topology on \( X \).

**Counter example:**
- Let \( X = \mathbb{R} \) and \( Y = \{0, 1\} \cup \{1/2\} \)

\( [1/2, 1/2] \) is open in \( Y \).

\( ((1/2, 1/2)) \) is open in \( \mathbb{R} \).

In fact, \([1/2, 1/2]\) is a basis element for subspace topology, but it is order topology every basis element containing 1 is of the form \((a, 1/2]\).
where \( a_1, a_2 \) = \{ y, y', a \} \), for some \( a \).

as \( a \) \in \mathbb{R}, \ a \in [0,1].

\[ \Rightarrow \ 0 < a < 1 \quad \text{or} \quad a + a < 1 \]

\[ \Rightarrow a \in (0,1) \]

\[ \Rightarrow a_{1,2} \neq \{x\} \]

\( \{x\} \) is not open in order topo.

Definition: \( X \)-ordered set and \( y \in X \). We say that \( y \) \in Convex if for any \( a, b \in X \), the open interval \( (a, b) \) is \( y \).

Obviously, every interval or open ray is convex.

Theorem: let \( X \) be an \( X \)-ordered set and \( y \in X \). Then, the \( \text{order topo on } y \) is same as the \( \text{topo on } y \) inherits as a subspace of \( X \).

Proof: We know that the collection of all open rays in \( X \) forms a subbasis for order topo on \( X \).

Let \( S_1 \) = \text{collection of open rays in } X \text{ intersecting with } y.

Then \( S_1 \) is subbasis for subspace topo on \( y \).

Let \( S_2 \) = \text{collection of all open rays in } y.

\( S_2 \) is subbasis for order topo on \( y \).

Let \( A \subseteq S \), \( A = \{a, a, a, \ldots\} \) or \( (a, \infty) \) \( A \) for some \( a \in X \).

Suppose that \( A = \{a, a, a, \ldots\} \)

if \( a \leq a \) then

\( A = \{a, a, a, \ldots\} \)

\( \Rightarrow A \) is an open ray in \( y \).

\( \Rightarrow a \in S_2 \)

\( \Rightarrow A \) is open in order topo.

if \( a \leq a \) then \( A = \{a, a, a, \ldots\} \) or \( (a, \infty) \) \( \Rightarrow A \) is open in order topo.

\( \Rightarrow A = \{a, a, a, \ldots\} \) or \( (a, \infty) \) \( \Rightarrow A \) is open in order topo.

\( \Rightarrow A = \{a, a, a, \ldots\} \) or \( (a, \infty) \) \( \Rightarrow A \) is open in order topo.

\( \Rightarrow A = \{a, a, a, \ldots\} \) or \( (a, \infty) \) \( \Rightarrow A \) is open in order topo.

\( \Rightarrow A \) is open in order topo.

Thus, in any case, \( A \) is open in order topo.
The subspace topology generated by $S_1$ is subset of order topology, i.e. subspace topology $\preceq$ order topology.

Conversely, every open ray in $y$ is the intersection of $y$ with an open ray in $x$.

Every element of $S_2$ is in $S_1$.

Order topology on $y \subseteq$ subspace topology on $y$.

From (1) and (2) we get:

Order topology on $y \subseteq$ subspace topology on $y$.

Show that if $y$ is a subspace of $x$ and $A$ is a subset of $y$,

then the topology $A$ inherits as a subspace of $y$ is the same as

the topology it inherits as a subspace of $x$.

Let $T_1, T_2$ be subspace topology on $A$ inherits from $y$ and $x$ respectively.

To show that $T_1 = T_2$.

Let $U \in T_1$.

$= U \cap N_A$ where $U$ is open in $y$.

But $y$ is a subspace of $x$.

$= \bigcup_{V \subseteq U}$ for some $V$ open in $x$.

$= \bigcup_{V \subseteq U}$ as $U \cap N_A$.

$= \bigcup_{V \subseteq U}$ as $U \cap N_A$.

$= T_2 \subseteq T_1$.

Let $U \in T_2$.

Then $U = V \cap N_A$ for some $V$ open in $x$.

$= V \cap (\text{Any}) = V$ (Any = $A$).

$= \bigcup_{V \subseteq U}$ where $V = V \cap N_A$, $V'$ open in $y$.

$= V \cap N_A \subseteq \gamma$.

$= \gamma \subseteq \gamma$.

$\Rightarrow \gamma \subseteq \gamma$.

From (1) and (2) we get:

$T_1 = T_2$. 
Ex. 2. If \( T \) and \( T' \) are topologies on \( X \), and \( T' \) is strictly finer than \( T \), what can you say about the corresponding subspace topologies on the subset \( Y \) of \( X \)?

Suppose that \( T' \) and \( T \) be subspace topology on \( Y \)

Then \( T_Y' = \{ U \cap Y \mid \text{U open in } T' \} \)

and \( T_Y = \{ U \cap Y \mid \text{U open in } T \} \)

Let \( U \in T_Y \)

\( \Rightarrow U = \bigcup_{i} \text{U}_i \)

\( \Rightarrow U \cap Y = \bigcup_{i} (U_i \cap Y) \)

\( \Rightarrow U \cap Y \in T_Y' \)

\( \Rightarrow U \in T_Y' \)

\( \Rightarrow T_Y \subseteq T_Y' \)

\( \Rightarrow T_Y' \) is finer than \( T_Y \)

\( T_Y \) need not be strictly finer than \( T_Y' \)

Counter example: Let \( X = \{ a, b, c \} \), \( Y = \{ \phi, x \} \)

Let \( T' = P(X) \), \( T = \{ \phi, x \} \)

Then \( T' \not\subseteq T \)

\( T_Y' = \{ \phi, Y \} \), \( T_Y = \{ \phi, x \} \)

\( \Rightarrow T_Y' \neq T_Y \)

\( \Rightarrow T_Y' \) is not strictly finer than \( T_Y \)

Ex. Consider the set \( Y = [0, 1] \) as a subspace of \( \mathbb{R} \). Which of the following sets are open in \( Y \)? Which are open in \( \mathbb{R} \)?

1) \( \{ x \in Y \mid x < 1 \} \)

\( \{ x \in Y \mid x < 1 \} = (-\infty, 1) \cap \mathbb{R} \)

\( \{ x \in Y \mid x < 1 \} \) is open in \( Y \) \( \left( \{ x \in \mathbb{R} \mid x < 1 \} \right) \) is open in \( \mathbb{R} \)
Similarly, \((-1, \frac{1}{2})\) is open in \(\mathbb{R}\) and \(Y\).

Their union is also open in \(Y\) and \(\mathbb{R}\):

\[ B = \{ x \in Y \mid \frac{1}{2} < x < 1 \} \]

\[ C = \{ x \in Y \mid \frac{1}{2} < x < 1 \} = (-1, \frac{1}{2}) \cup \left(\frac{1}{2}, 1\right) \]

For any open set \(U\) in \(\mathbb{R}\),

\[ \left(\frac{1}{2}, 1\right) \cup U \not\subseteq Y \]

\[ \Rightarrow \left(\frac{1}{2}, 1\right) \text{ is not open in } Y \]

Similarly, \([-1, \frac{1}{2}]\) not open in \(Y\).

\[ B \not\subseteq Y \text{ is not open in } Y \]

Also, \(B\) is not open in \(\mathbb{R}\).

\[ C = \{ x \in Y \mid \frac{1}{2} < x < 1 \} = [-1, \frac{1}{2}] \cup (\frac{1}{2}, 1) \]

\[ D = \{ x \mid \frac{1}{2} < x < 1 \} \]

Do neither open in \(Y\) nor open in \(\mathbb{R}\).

\[ E = \{ x \mid 0 < x < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}_+ \} \]

\[ E = \{ x \mid 0 < x < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}_+ \} \]

\[ = \left(\frac{1}{10}, \frac{1}{2}\right) \cup (0, 1) \]

\[ = \left(\frac{1}{10}, \frac{1}{2}\right) \cup (0, 1) \]

\[ = \frac{1}{x} \notin \mathbb{Z}_+ \Rightarrow \frac{1}{x} + n \text{ for any } n \in \mathbb{Z}_+ \]

\[ \Rightarrow x + \frac{1}{n} \notin \mathbb{Z}_+ \]

\[ \Rightarrow x \notin (0, 1) \]

\[ \Rightarrow x \in (0, 1) - \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\} \]

\([-1, 0)\) (car) \(\cap \mathbb{R}^2\) is open in \(\mathbb{R}^2\).

Let \(k = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}\)

Check \(x \in (0, 1) - K\) open in \(\mathbb{R}\)?

Let \(x \in (0, 1) - K\)

\[ \Rightarrow \exists x \in (0, 1) - \frac{1}{n} \text{ for any } n \in \mathbb{Z}_+ \]

\]
\[ 70 \text{ and it is not an integer} \]

\[ n \text{ such that } n \leq \frac{1}{x} < n+1 \]

\[ \frac{1}{n} \leq x \leq \frac{1}{n+1} \]

Choose \( \frac{1}{n} \leq x \leq \frac{1}{n+1} \)

\[ x \in (\frac{1}{n}, \frac{1}{n+1}) \subseteq (\frac{1}{n+1}, \frac{1}{n}) \subseteq (0,1) - K \]

\( (0,1) - K \text{ is open in } \mathbb{R} \).

\[ E = (-1,0) \cup (0,1) - K \text{ is open in } \mathbb{R} \]

Now \( E = E \cap Y \)

\[ E \text{ is open in } Y. \]

*Closed sets and limit points*:

Defn: A subset \( A \) of a topological space \( X \) is said to be closed if the set \( X - A \) is open.

Examples:

1. The subset \( (-1,1) \) of \( \mathbb{R} \) is closed because its complement \( \mathbb{R} - (-1,1) = (-\infty, -1) \cup (1, \infty) \) is open in \( \mathbb{R} \).

2. In the plane \( \mathbb{R}^2 \), the set \( F = \{(x,y) | x \geq 0 \text{ and } y \geq 0\} \) is closed because its complement is the union of the two sets \((-\infty, 0) \times \mathbb{R} \) and \( \mathbb{R} \times (-\infty, 0) \).
3. In a discrete topological space on a set $X$, every set is open; it follows that every set is closed as well.

... every subset of $X$ is both open and closed in $X$.

4. Let $Y = [2, 4) U (3, 6) \subseteq \mathbb{R}$.

\[ [2, 4] = (0.5, 2.5) \cap Y \]

open in $\mathbb{R}$.

\[ (3, 4) = (3, 4) \cap Y \]

open in $\mathbb{R}$.

\[ (3, 4) \text{ is open in } Y. \]

\[ (3, 4) \cap (3, 4) = [1, 2] \text{ is open in } Y. \]

\[ Y \setminus [1, 2] = (3, 4) \text{ is closed in } Y. \]

Theorem: Let $X$ be a topological space. The following conditions hold:

1. $\emptyset$ and $X$ are closed.

2. Arbitrary intersections of closed sets are closed.

3. Finite unions of closed sets are closed.

Proof: 3) Since $\emptyset$ and $X$ are open in $X$.

Their complements $X - \emptyset = X$ and $X - X = \emptyset$ are open in $X$.

$\emptyset$ and $X$ are closed in $X$.

2) Let $\{A_{\alpha}\}$ be a collection of closed sets.

Let $A = \bigcap A_{\alpha}$.

$x - A = X - \bigcap A_{\alpha} = \bigcup (x - A_{\alpha})$ open in $X$.

... $x - A_{\alpha}$ is open in $X$ for each $\alpha$.

because $A_{\alpha}$ is closed in $X$.

and we know that arbitrary unions of open sets are open.
\[ \forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } x \in A \implies d(x, A) < \epsilon \]

3. Let \( A_1, A_2, \ldots, A_n \) be closed in \( X \).

Let \( A = A_1 \cup A_2 \cup \cdots \cup A_n \) closed in \( X \) where \( A_1, A_2, \ldots, A_n \) are closed in \( X \).

Let \( A_1, A_2, \ldots, A_n \) be closed in \( X \).

To show: \( A = A_1 \cup A_2 \cup \cdots \cup A_n \) is closed in \( X \).

\[ X - A = \{ x \in X : x \notin A \} = (X - A_1) \cap (X - A_2) \cap \cdots \cap (X - A_n) \]

Recall that \( X - A \) is open in \( X \) \( (\because X - A_i \) open in \( X \) for \( i \in \mathbb{N} ) \)

\[ \implies X - A \text{ is open in } X \]

\[ \implies A \text{ is closed in } X \]

\[ \text{because } A = (X - A) \text{ is closed. } \]

Note: Arbitrary unions of closed sets need not be closed.

Counter Example:

\[ F_\alpha = \left[ \frac{1}{\alpha}, \frac{1}{2} \right], \quad \alpha = 2, 3, 4, \ldots \]

Each \( F_\alpha \) is closed in \( \mathbb{R} \).

But \( \bigcup F_\alpha = (0, 1) \) is not closed in \( \mathbb{R} \), because its complement \( (0, 1) \cup \{ \alpha \} \) is not open in \( \mathbb{R} \).

Theorem: Let \( Y \) be a subspace of \( X \). Then a set \( A \) is closed in \( Y \) if and only if it equals the intersection of a closed set of \( X \) with \( Y \).

Proof: Let \( A \in Y \).

Suppose that \( A \) is closed in \( Y \).

\[ \implies Y - A \text{ is open in } Y \]

\[ \implies Y - A = U_Y \text{ for some } U \text{ open in } X \]

\[ \implies X - A \text{ is closed in } X \]
\[(x-y) \cap Y = \emptyset \quad \text{Suppose that } A \text{ is closed in } Y.\]

Closed in \(X\) \(A = (x-y) \cap Y = \emptyset \) where \(C \) is closed in \(X\).

So that \((x-C) \cap Y \) is open in \(Y\).

But \((x-C) \cap Y = Y - A = \emptyset C\).

Hence \(Y - A \) is open in \(Y\). Complement of \(A\).

\(\Rightarrow \quad A \text{ is closed in } Y.\)

Conversely suppose that \(A \) is closed in \(Y\).

Then \(Y-A \) is open in \(Y\).

So by definition it equals the intersection of an open set \(U \) of \(X\) with \(Y\).

The set \((X-U) \) is closed in \(X\) and \(A = Y - (X-U)\).

So that \(A \) equals the intersection of a closed set of \(X\) with \(Y\).

**Theorem:** Let \(Y\) be a subspace of \(X\) and \(A \subseteq Y\). Then,

(a) If \(A \) is closed in \(X\) then \(A \) is closed in \(Y\).

(b) \(Y\) closed in \(X\) and \(A \) is closed in \(Y\) then \(A \) is closed in \(X\).

**Proof:**

(a) \(A = \emptyset \cap Y \quad (: A \subseteq Y)\) closed in \(X\).

\(\Rightarrow \quad A \) is closed in \(Y\).

(b) Given \(A \) is closed in \(Y\).

\(A = F \cap Y \) for some \(F \) is closed in \(X\).

Since \(F \) and \(Y \) both are closed in \(X\).

\(\Rightarrow \quad F \cap Y \) closed in \(X\).

\(\Rightarrow \quad A \text{ is closed in } X.\)

\(\text{Closure and Interior of a Set:}\)

**Definition:** Let \(A\) be a subset of a topological space \(X\). Interior of \(A\) is denoted by \(\text{Int}(A)\) and is defined as the union of all open sets contained in \(A\) or union of all open subsets of \(A\).

\(\text{Int}(A) \) is largest open subset of \(A\).

Also \(\text{Int} \ A = A \) if \(A\) is open.
Def. Let $A$ be a subset of the topological space $X$. The closure of $A$ is denoted by $\overline{A}$ and is defined as the intersection of all closed sets containing $A$. If $A$ is the smallest closed set in $X$ containing $A$, then $\overline{A} = A$. Iff $A$ is closed.

Theorem. Let $Y$ be a subspace of $X$, let $A$ be a subset of $Y$. Let $\overline{A}$ denote the closure of $A$ in $X$. Then the closure of $A$ in $Y$, $\overline{A}_Y$, equals $\overline{A}_Y$.

Proof: Let $B = \overline{A}$ in $Y$.

Wrt: $B = \overline{A}_Y$.

Since $B$ is closed in $Y$ ($\overline{A}$ is closed in $X$, then $\overline{A}_Y$ is closed in $Y$)

$B = \overline{A}_Y$ for some $F$ is closed in $X$.

By $\overline{A} = \overline{A}_Y = \overline{A}_Y = \overline{A}_Y$.

$A \subseteq \overline{A}_Y$ (since $A$ is intersection of all closed sets containing $A$).

$\overline{A}_Y \subseteq F$ for some $F$ is closed in $X$.

$\overline{A}_Y \subseteq F$ (since $\overline{A} = \overline{A}_Y$)

From equations 1 and 2, we get $B = \overline{A}_Y$.

Theorem: Let $A$ be a subset of the topological space $X$.

a) The $x \in A$ if every open set $U$ containing $x$ intersects $A$.

b) Supposing the topology of $X$ is given by a basis, then $x \in A$ if every basis element $B$ containing $x$ intersects $A$.

Proof: a) We prove that $x \in A$ if $x$ and a nbd. $U$ of $x$ such that $U \cap A = \emptyset$.

Assume that $x \notin A$.

Since $A$ is closed.
Let $A = (0, 1) \subseteq \mathbb{R}$.

Then $A = [0, 1] U (1/2, 1]$.

Let $A = (0, 1/2) \subseteq \mathbb{R}$.

Then $A = [0, 1/2] U (1/2, 1]$.

Consider the subspace $Y = (0, 1)$ of the real line $\mathbb{R}$.

Let $A = (0, 1/2)$ be a subset of $Y$.

Then $\overline{A} = [0, 1/2]$.

But $\overline{A} \cap Y = (0, 1/2)$. 

$Y = (0, 1)$ is a subspace of $\mathbb{R}$. 

Let $A = (0, 1)$. 

Then $\overline{A} = [0, 1]$.

So $A \subseteq Y \subseteq \mathbb{R}$.
**Limit Point:** Let $A$ be a subset of topological space $X$. An element $x \in X$ is called a limit point of $A$ if every nbd of $x$ contains a member of $A$ other than $x$. If $x$ is limit point of $A$ then $(\forall U \text{ nbd of } x \ L(U \cap A) \neq \phi)$

Note: Set of limit points of $A$ is denoted by $A'$.

**Examples:**

1. Let $A = (0, 1] \subset \mathbb{R}$ then $A' = [0, 1]

2. If $A = \{1, 2, 3\} \subset \{1, 2, 3\}$ then $A' = \emptyset$

3. If $A = \{1, 2, 3\} \subset \mathbb{R}$ then $A' = \mathbb{R}$

4. Let $A = (0, 1) \subset \mathbb{R}$ then $A' = \mathbb{R}$

5. $\lim A^c A = \mathbb{Q}$ hence $A' = \mathbb{R}$

6. $A = \mathbb{R}^+$ \subset \(0, \infty) \) then $A' = (0, \infty)$

**Theorem:** Let $A$ be a subset of the topological space $X$. Let $A'$ be the set of all limit points of $A$. Then $\bar{A} = A \cup A'$

**Proof:** Let $x \in A \cup A'$

If $x \in A$ then $x \in A'$ (since $A \subseteq A'$)

If $x \in A'$ then for every nbd $U$ of $x$, $U \cap (A - \{x\}) \neq \emptyset$ implies $A \cap U \neq \emptyset$

If $x \in A'$ then for every nbd $U$ of $x$, $U \cap (A - \{x\}) \neq \emptyset$

Suppose $x \in A'$. Let $U$ be any nbd of $x$.

**1:** 

- $U \cap (A - \{x\}) \neq \emptyset$

**2:** 

- $A \cap U \neq \emptyset$

Correctly let $x \in A$

**From eqs. 1 & 2**

- $A \in A'$

Thus $x \in A'$. Therefore $\bar{A} = A \cup A'$. 

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Corollary: A subset of a topological space is closed iff it contains all its limit points.

Proof: \( A \) is closed iff \( A = \overline{A} \) iff \( A = A \cup A^c \) (\( A = A \cup A^c \))

- Hausdorff spaces:
  \( \text{Let } X = \{a, b, c\}; \quad \gamma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\} \)

Then \( \gamma \) is topology on \( X \):
- \( \{b\} \) is not closed in \( X \) because \( X \setminus \{b\} = \{a, c\} \notin \gamma \)
- \( \{a, c\} \) is not open in \( X \).

- Def: Let \( X \) be a topological space. \( \{x_n\} \) be a sequence of elements of \( X \). We say that \( x_n \to x \) in \( X \) if for every \( U \) of \( X \in \mathbb{N} \) such that \( x_n \in U \), \( n \notin \mathbb{N} \).

Ex: Let \( A, B \) and \( A_n \) denote subsets of a space \( X \). Prove the following.

1. If \( A \subseteq B \) then \( A \subseteq B \).
2. \( A \cup B = \overline{A \cup B} \).
3. \( (A_n \supseteq \cup A_n) \) give an example where equality fails.

Solution: Let \( X = A \). Let \( U \) be an open of \( X \).
- \( U \) is \( x_n \in \cap U \).
- \( \rightarrow x_n \in x_n \) let \( U \notin \text{Interior of } X \).

\( \rightarrow A \subseteq B \). Let \( x \notin A \).
- \( \rightarrow x \notin B \).

Assume that \( x \notin \overline{A \cup B} \). Let \( x \notin A \).
- \( \rightarrow x \notin B \).

If \( x \in U \) such that \( U \cap A = \emptyset \).
- \( \rightarrow x \in U \).
- \( \rightarrow \emptyset \)

If \( x \in U \) such that \( U \cap A = \emptyset \).
- \( \rightarrow (U \cap A \cap B) = \emptyset \)
- \( \rightarrow x \notin A \cup B \).
- \( \rightarrow x \notin A \cup B \).
- \( \rightarrow x \notin A \cup B \).
The converse is proved in (2). 

5. \( \mathbf{U}_A \cap \mathbf{U}_A \) 

Let \( x \in \mathbf{U}_A \) 

For some \( a \) 

Let \( U \) be any set of \( A \) such that 

Then \( \mathbf{U}_A \cap \mathbf{U}_A = \mathbf{U}_A \left( \mathbf{U}_A \right) = \phi \) 

\( \because A_x \subseteq U_A \) 

Equality does not hold. 

Counter Example 

Let \( A_y = \left[ \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right], \ y \neq x \) 

\( A_y = A_y \) 

\( U_A = U_A = (o_1) \) 

\( U_A = (o_1) \) 

\( U_A = (o_1) \) 

\( U_A = (o_1) \) 

\( U_A \neq U_A \) 

52. \( \overline{A} \cap B = \overline{A} \cap B \) 

\( \overline{A} \cap B = \overline{A} \cap B \) 

\( \text{Justify} \) 

\( \text{Counter example:} \) 

\( A = (o_1), \ b = (1, 2) \) 

\( \overline{A} \cap B = \phi \) 

Then \( \overline{A} \cap B = \phi \) 

and \( \overline{A} \cap B = (o_1) \) 

\( \overline{A} \cap B = \phi \) 

But \( \overline{A} \cap B \neq \overline{A} \cap B \) 

53. \( \overline{A} - B = \overline{A} - B \) 

\( \text{Justify} \) 

\( \text{Counter example:} \) 

Let \( A = (1, 1), \ b = (1, 2) \) 

\( A - B = (1, 1) - (1, 2) = (1, 1) \) 

and \( A - B = (1, 1) \) 

\( \Rightarrow A - B = (1, 1) = (1, 1) \) 

\( \Rightarrow A - B = (1, 1) = (1, 1) \) 

\( \Rightarrow A - B = (1, 1) \) 

\( \Rightarrow A - B = (1, 1) \)
By previous Example (Prop 44).

Let \( x_n = \frac{1}{n}, y_n = 1, n \geq 1. \)

\( \{y_n\} \) is a constant sequence.

i) Let \( U \) be a nbh of \( b. \) Then \( b \in U \)

\[ x_n \in U \Rightarrow n = 1 \Rightarrow x_1 = 1 \]

ii) Let \( U \) be a nbh of \( a. \)

\[ U \] is an open set containing \( a. \)

\[ U = x \subset d,b \]

In both cases \( b \in U \)

\[ x_n \in U, \quad \forall n \quad (\text{in } n = 1 \text{ here}) \]

\[ x_n \to a. \]

Similarly \( x_n \to c. \)

**Definition:** A topological space \( X \) is called a Hausdorff space if for each pair \( x_1, x_2 \) of distinct points of \( X \) there exist neighborhoods \( U_1 \) and \( U_2 \) of \( x_1 \) and \( x_2 \) respectively, that are disjoint.

**Theorem:** Every finite point set in a Hausdorff space is closed.

**Proof:** It is sufficient to prove that every one point set or singletons \( \{x_0\} \) is closed in \( X. \)

Let \( x \neq x_0 \) as \( X \) is Hausdorff space. \( \exists \) nbh \( U \) and \( V \) of \( x \) and \( x_0 \) respectively s.t. \( UV = \emptyset. \)

Thus \( x \in U \)

\[ U \cap \{x_0\} = \emptyset \]

\[ x \neq x_0 \]

\[ \{x_0\} \subseteq \{x_0\} \]

\[ \{x_0\} \subset \{x_0\} \quad (\because \{x_0\} \subset \{x_0\}) \]
Theorem: Let $X$ be a space satisfying the $T_i$ axiom; let $A$ be a subset of $X$. Then $x$ is a limit point of $A$ if every nbhd of $x$ contains infinitely many points of $\overline{A}$.

Proof: Suppose that every nbhd $U$ of $x$ contains infinitely many points of $\overline{A}$.

Let $V_n = \bigcup (A \setminus x_n)$. Then $V_n$ is a finite set.

Let $V_n = \bigcup (A \setminus x_n) = x_1, x_2, \ldots, x_n$.

Since $X$ is a $T_i$ space,

$$V_n(x_1, x_2, \ldots, x_n)$$

is closed.

$$x_1, x_2, \ldots, x_n$$

is open in $X$.

$$\bigcup (x_1, x_2, \ldots, x_n)$$

is open in $X$.

Note that $x \in U$ and $x \in \bigcup (x_1, x_2, \ldots, x_n)$.
$V = \bigcup_{n \in \mathbb{N}} (x - \frac{1}{n}, x + \frac{1}{n})$ is open in $X$.

This $V$ is a neighborhood of $x$ such that $\bigcup_{n \in \mathbb{N}} (x - \frac{1}{n}, x + \frac{1}{n}) = \emptyset$.

Which is a contradiction to the fact that $x$ is a limit point.

Our assumption was wrong.

Every neighborhood of $x$ contains infinitely many points of $A$.

**Theorem:** If $X$ is a Hausdorff space, then a sequence of points of $X$ converges to at most one point of $X$.

**Proof:** Suppose that $x_n \rightarrow x$. Assume that $x \neq y$.

There exist distinct neighborhoods $U$ and $V$ of $x$ and $y$, respectively.

Let $N$ such that $x_n \in U$, $v \in V$.

Similarly, $N_1$ such that $x_n \in U$, $v \in V$.

Let $N = \max \{ N, N_1 \}$.

$\Rightarrow x_n \in U \cap V$ and $v \in V$.

$\Rightarrow$ a contradiction.

$x = y$.

**Exercise:** Show that a subspace of a Hausdorff space is Hausdorff.

**Solution:** Let $X$ be a Hausdorff space.

Let $Y$ be a subspace of $X$.

Let $x, y \in Y$.

As $x \in X$, $\exists U \cap V$ open in $X$ such that $x \in U \cap V$.

And $U \cap V = \emptyset$.

Let $U = \bigcup_{n \in \mathbb{N}} (x - \frac{1}{n}, x + \frac{1}{n})$ and $V = \bigcup_{n \in \mathbb{N}} (y - \frac{1}{n}, y + \frac{1}{n})$.

Then $U$ and $V$ are open in $Y$.

Also, $x \in U \cap V$.

And $U \cap V = (\bigcup_{n \in \mathbb{N}} (x - \frac{1}{n}, x + \frac{1}{n})) \cap \bigcup_{n \in \mathbb{N}} (y - \frac{1}{n}, y + \frac{1}{n}) = \emptyset$.

Thus $Y$ is a Hausdorff space.

The subspace of a Hausdorff space is Hausdorff.
Ex. Show that every order topology is Hausdorff.

Set: Let $X$ have order topology

Test: $X$ is Hausdorff.

Let $x, y \in X$.

Without loss of generality, we assume $x < y$.

If $x, y \in X$ then $x \in (0, y)$, $y \in (x, 0)$

Paths $(0, x, y)$ and $(x, 0, y)$ are open in $X$, their joint $f$ contains $x$ and $y$ respectively.

Suppose $z \in X$ is between $x$ and $y$.

Then $x \in (0, z)$, $y \in (z, 0)$

Note that $C(0, z) \cap (z, 0) = \emptyset$.

$X$ is Hausdorff.

Ex. In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = \frac{1}{n}$ converge?

Solution: If $x_n \to x$ then $x_n$ is finite or $x_n = x$.

Let $x \in \mathbb{R}$, we claim that $\frac{1}{n} \to x$.

Let $U$ be a neighborhood of $x$.

$\implies x \in U = U^c_\emptyset$

As $U$ is open, $x \in U$ is finite.

$\implies \exists \emptyset \in U \text{ is finite}.

Ex. Let $\{x_n\}$ in $X$, Hausdorff. If the diagonal $\Delta$ is closed, is $\Delta = \{x \mid x = x_n\}$ closed in $X \\times X$?

Solution: Suppose $x \in X$.

To prove $\Delta$ is closed in $X \\times X$.

Let $x_n \in \Delta$

To show $x_n \in \Delta$.

If it is enough to show $y = x_n$.

Assume that $y \neq x$.

As $X$ is Hausdorff,

$\exists U \text{ and } V \text{ of } x$ and $y$ respectively, s.t. $U \cap V = \emptyset$.
Also \( x, y \in U \cup V \implies U \cup V \) is a nbhd of \( x, y \).

Also \( x, y \in A \)

\( (U \cup V) \cap A \neq \emptyset \)

Let \( a, b \in (U \cup V) \cap A \)

\( a, b \in U \cup V \) and \( a, b \in A \)

\( a \in U \cup V \) and \( b \in U \cup V \)

\( a \in U \cup V \)

Contradiction \( \emptyset \)

\( a = b \)

\( a \in U \cup V \)

From \( \emptyset \) and \( a \in U \cup V \)

\( a = b \)

\( a \in U \cup V \)

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\( a \in U \cup V \)
If for every \( V \) is open in \( y, \ f^{-1}(y) \) is open in \( x \).

To prove that for functions \( f: \mathbb{R} \to \mathbb{R} \), the \( \varepsilon-\delta \) definition of continuity implies the open set definition (conversely).

**Solution:** Assume that \( f \) is continuous in the sense of \( \varepsilon-\delta \) definition.

Let \( V \) be an open set in \( \mathbb{R} \) (co-domain).

To show that \( f^{-1}(x) \) is open in \( \mathbb{R} \) (co-domain).

Suppose that \( f^{-1}(x) = \emptyset \), then obviously \( f^{-1}(x) \) is open in \( \mathbb{R} \).

Consider \( x \in f^{-1}(x) \).

Let \( x \in f^{-1}(x) \).

Then there exists \( \varepsilon > 0 \) such that \( (x-\varepsilon, x+\varepsilon) \subseteq f^{-1}(x) \).

Now \( \exists \delta > 0 \) such that \( |x-x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \).

Choose \( \varepsilon = \min \{f(x_0) - a, b - f(x_0)\} \).

Then for \( x \in (x_0 - \delta, x_0 + \delta) \), \( f(x) \in f^{-1}(x) \).

Thus \( (x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(x) \).

Choose \( x_0 \in (x_0 - \delta, x_0 + \delta) \), \( f(x_0) \in f^{-1}(x) \).

Choose \( \delta = \min \{f(x_0) - a, b - f(x_0)\} \).

Then \( (x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(x) \).

Consequently assume that \( f \) is continuous in the sense of open set definition.

Let \( x \in \mathbb{R} \). Let \( \varepsilon > 0 \) want \( \delta > 0 \) s.t.

\[ |x-x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \]

We have \( f(x_0) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \).

Since \( V \) is open \( \Rightarrow f^{-1}(x) \) is open.

Also \( x \in f^{-1}(x) \).

\[ (a, b) \cup x \in (a, b) \cap f^{-1}(x) \]

Choose \( \varepsilon = \min \{f(x_0) - a, b - f(x_0)\} \).

Then \( (x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(x) \).

\[ |x-x_0| < \delta \Rightarrow x \in (x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(x) \].
\[
\begin{align*}
\rightarrow \quad & f(A) \in V \Rightarrow (f(x)) \in V, \quad \forall x \in A \\
\quad & \left( f(x) \right) \in V \Leftrightarrow x \in f^{-1}(V) \\
\begin{cases}
\text{(i)} \quad \text{Let } f : X \rightarrow \mathbb{R} \quad \text{be the identity function, } f(x) = x, \quad x \in \mathbb{R} \\
\text{then } f \text{ is continuous.}
\end{cases}
\end{align*}
\]

**Theorem:** Let \( X \) and \( Y \) be topological spaces. Let \( f : X \rightarrow Y \).

Then the following are equivalent:

1. \( f \) is continuous.
2. For every subset \( A \) of \( X \), one has \( f(A) \subset f(A) \).
3. For every closed set \( B \) of \( Y \), the set \( f^{-1}(B) \) is closed in \( X \).
4. For each \( x \in X \) and each open \( V \) of \( f(x) \), there is a \( U \) of \( x \) such that \( f(U) \subset V \).

**Proof:** We show that:

1. \( \Rightarrow (\Rightarrow 2) \Rightarrow 3 \Rightarrow 4 \Rightarrow 1 \).
2. \( \Rightarrow 2 \).

To show that \( f(A) \subset f(A) \).

Let \( f(A) \in f(A) \).

We have \( x \in A \).

Let \( U \) be any neighborhood of \( f(x) \).

\( f(U) \) is open in \( X \) and \( x \in f(U) \).

As \( x \in A \):

\( f(U) \ni A \neq \emptyset \)

Let \( x \in f(U) \cap A \).

\( f(x) \in U \), \( f(x) \in f(A) \).

\( f(A) \subset f(A) \).

Thus \( f(A) \subset f(A) \).

2. \( \Rightarrow 3 \).

Let \( B \) be closed in \( Y \).

To show that \( f(B) \) is closed in \( X \).

Let \( A = f^{-1}(B) \).
To show that $\overline{A} = A$ obviously $A \subseteq \overline{A}$ — (1)

It remains to show that $\overline{A} \subseteq A$.

Let $x \in \overline{A}$, i.e., $f(x) \in f(A) \subseteq f(A) \cup \{f(a) \mid a \in A \} = B$

Thus $x \in f^{-1}(B) \Rightarrow f(x) \in B$

$\Rightarrow x \in f^{-1}(B) = A$. Thue $x \subseteq \overline{A} \Rightarrow x \in A$

$\Rightarrow A \subseteq \overline{A}$ — (2)

From eqns (1) and (2) we get $\overline{A} = A$

Thus $A$ is closed in $X$.

i.e. $f^{-1}(B)$ is closed in $X$.

3). $\Rightarrow$ 2).

Let $V$ be an open set of $Y$.

To show that $f^{-1}(V)$ is open in $X$.

Let $B = Y - V$

Then $f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = f^{-1}(V)$.

Since $B$ is closed in $Y$.

Then $f^{-1}(B)$ is closed in $X$. (By hypothesis)

$\Rightarrow f^{-1}(V)$ is closed in $X$.

$\Rightarrow f^{-1}(V)$ is open in $X$.

Thus $f$ is continuous.

3). $\Rightarrow$ 4).

Let $x \in X$ and let $V$ be a nbhd of $f(x)$. Then the set $U = f^{-1}(V)$ is a nbhd of $x$. Set $f(U)$ c. $V$.

4). $\Rightarrow$ 3).

Let $V$ be open in $Y$.

Let $x$ be a point of $f^{-1}(V)$

Then $f(x) \in V$. So by hypothesis, there is a nbhd $U_x$ of $x$ s.t. $f(U_x) \subseteq V$. Then $U_x \subseteq f^{-1}(V)$.

It follows that $f(U_x)$ can be written as the union of the open sets $U_x$, so that it is open.

\[ \therefore \]
- **Homeomorphism**: Let $X$ and $Y$ be topological spaces. Let $f: X \rightarrow Y$ be a homeomorphism if $f$ is continuous, one-to-one, onto, and $f^{-1}: Y \rightarrow X$ is continuous. $X$ and $Y$ are homeomorphic if $f$ is a homeomorphism between them.

**Theorem**: $f: X \rightarrow Y$ bijective, continuous.

Then the following are equivalent:

1. $f^{-1}: Y \rightarrow X$ continuous.
2. $f$ is an open map.
3. $f$ is a closed map.

**Proof**: $1 \Rightarrow 2$

Let $A$ is open in $X$.

$$f^{-1}(f(A)) = f^{-1}(B)$$

$A$ is open in $X$ and $f^{-1}: Y \rightarrow X$ is continuous.

$(f^{-1})^{-1}(A)$ is open in $Y$.

$e \in (f^{-1})^{-1}(A)$ is open in $Y$.

$\Rightarrow f$ is an open map.

$2 \Rightarrow 3$

Let $A$ is closed in $X$.

$X-A$ is open in $X$.

$f(X-A)$ is open in $Y$.

$f(A)$ is closed in $Y$.

$\Rightarrow f^{-1}(f(A))$ is open in $Y$.

$f^{-1}(f(A)) = f^{-1}(B)$ is open in $Y$.

$\Rightarrow f^{-1}$ is continuous.

**Ex. 7.5**: [a,b] is homeomorphic to $[0,1]$. Justify.

**Sol.** Yes.

$f: [0,1] \rightarrow [a,b]$ by $f(t) = (1-t)a + tb$

$= a + (b-a)t$
\[ f(0) = a, f(1) = b. \]

* f is continuous.

Let \( a + (b-a)t = y \)

\[ t = \frac{y-a}{b-a} \]

Defined \( g : [a, b] \rightarrow [a, b] \) by \( g(y) = a + (b-a)y \).

\[ g \) is continuous.

(1) \( f \) is one-one iff \( \exists \, \exists \, g : y \rightarrow x \) st \( g \circ f = I_x \) where \( I_x \, x \rightarrow x \), here \( g \) is called left inverse of \( f \).

(2) \( f \) is onto iff \( \exists \, \exists \, g : y \rightarrow x \) st \( f \circ g = I_y \) where \( I_y \, y \rightarrow y \).

Here \( g \) is called right inverse of \( f \).

\( \Rightarrow \) \( f \) is bijective iff \( \exists \, \exists \, g : y \rightarrow x \) st \( f \circ g = I_y \) and \( g \circ f = I_x \).

\[ f \circ g(y) = f(g(y)) = f(a + (b-a)y) = a + (b-a)(y-a) = y \]

\[ = I_y \, [a, b] \]

\[ g \circ f(t) = g(a + (b-a)t) = a + (b-a)(a + (b-a)t) = a + (b-a)^2t - a \]

\[ = I_{\{t\}} \, [a, b] \]

\( \Rightarrow \) \( f \) is left and right inverse of \( f \).

\( \Rightarrow \) \( f \) is one-one and onto.

\( \Rightarrow \) \( f \) is bijective.

\( \Rightarrow \) \( f \) is one-one, onto, continuous and \( f^{-1} \) continuous.

\( \Rightarrow \) \( f \) is homeomorphic from \( \mathbb{R} \) to \( \mathbb{R} \).

Ex. Is \( (0,1) \) homeomorphic to \( (a, b) \)? Justify \( (\text{yes, some map}) \).

Ex. Is \( (0,1) \) homeomorphic to \( \mathbb{R} \)?

We: \( \lim f\left( \frac{\pi}{2} \right) = \infty \), \( \text{such that} \)

\[ f(x) = \tan x \]

as \( x \rightarrow \frac{\pi}{2} \), \( f\left( \frac{\pi}{2} \right) = \tan\left( \frac{\pi}{2} \right) = \infty \)

\[ \text{at} \, x = \frac{\pi}{2}, \, f(x) = \tan\frac{\pi}{2} = \infty \]

\( \Rightarrow \) \( f \) is discontinuous.\]
Let \( g : \mathbb{R} \rightarrow (\frac{-\pi}{2}, \frac{\pi}{2}) \) by \( g(y) = \tan y \). 

- \( g \) is continuous

\[ f \circ g(y) = f [g(y)] = f (\tan y) = y \]

and \( g \circ f(x) = g [f(x)] = g (\cot x) = x \)

\( \Rightarrow f \) is one-one and onto

\( \Rightarrow f \) is homeomorphism from \( (\frac{-\pi}{2}, \frac{\pi}{2}) \) to \( \mathbb{R} \).

Let \( h : (0, 1) \rightarrow (\frac{-\pi}{2}, \frac{\pi}{2}) \) by
\[ h(x) = (1-x)(\frac{-\pi}{2}) + x(\frac{\pi}{2}) \]

\( \Rightarrow \) Then \( h \) is homeomorphism from \( (0, 1) \) to \( (\frac{-\pi}{2}, \frac{\pi}{2}) \).

Let \( k : (0, 1) \rightarrow \mathbb{R} \) such that \( k(1) = \frac{\pi}{2} \)

\( \Rightarrow k \) is homeomorphism from \( (0, 1) \) to \( \mathbb{R} \).

Let \( s^1 = \left\{ (x, y) \mid x^2 + y^2 = 1 \right\} \subseteq \mathbb{R}^2 \)

Let \( f : (0, 1) \rightarrow s^1 \) s.t. \( f(\pi) = e^{2\pi i} = (\cos 2\pi, \sin 2\pi) \)

\( \frac{\pi}{2} \approx 1 \)

- \( f \) is 1-1 onto and continuous

\( \Rightarrow f \) is continuous.

It is enough to show that \( f \) is not open map.

Let \( U = (0, \frac{\pi}{4}) \cap (\frac{-\pi}{2}, \frac{\pi}{2}) \)

\( \Rightarrow U \) is open in \( (0, 1) \)

\( \Rightarrow U \) is open in \( \mathbb{R} \)

\( \Rightarrow f(U) = \) is not open in \( s^1 \) because \( f(U) \cap V \neq f(U) \) for any open set in \( \mathbb{R}^2 \)

\( \Rightarrow f \) is not open map

\( \Rightarrow f \) is not continuous

\( \Rightarrow f \) is not homeomorphism.

Q: Is \( (0, 1) \) homeomorphic to \( s^1 \)? (No).
Rules for Constructing Continuous Functions.

Theorem: Let \( X, Y \) and \( Z \) be topological spaces.

1. (Constant Function). If \( f: X \to Y \) maps all of \( X \) into the single point \( y_0 \) of \( Y \), then \( f \) is continuous.

2. (Inclusion). If \( A \) is a subspace of \( X \), the inclusion function \( i: A \to X \) is continuous.

3. (Composites) If \( f: X \to Y \) and \( g: Y \to Z \) are continuous, then the map \( g \circ f: X \to Z \) is continuous.

4. (Restricting the domain) If \( f: X \to Y \) is continuous and \( A \) is a subspace of \( X \), then the restricted function \( f\mid A : A \to Y \) is continuous.

5. (Restricting or expanding the range). Let \( f: X \to Y \) be continuous. If \( Z \) is a subspace of \( Y \), containing the image set \( f(X) \), then the function \( g: X \to Z \) obtained by restricting the range of \( f \) is continuous. If \( Z \) is a space having \( Y \) as a subspace, then the function \( h: X \to Z \) obtained by expanding the range of \( f \) is continuous.

6. (Local formulation of continuity). The map \( f: X \to Y \) is continuous if \( X \) can be written as the union of open sets \( U_1 \) such that \( f \mid U \) is continuous for each \( U \).

Proof: 1). Let \( f: X \to Y \). Let \( y \in Y \) and \( V \) be an open set in \( Y \). \( f^{-1}(y) = \emptyset \) if \( y \notin f(X) \).

2). Let \( V \) be an open set in \( Y \). Let \( y \in f(X) \), then \( f^{-1}(y) = \emptyset \) if \( y \notin f(X) \).

3). Let \( V \) be an open set in \( X \). Since \( f \) is continuous, \( f^{-1}(y) \) is open in \( X \).

4). Let \( i: A \to X \). Then \( i\mid A \) is a continuous function.
c). Let \( \text{gof}: x \rightarrow z \)

Let \( \nu \) be open in \( z \).

\( \Rightarrow g(\nu) \) is open in \( y \).

\( \Rightarrow g \circ f^{-1}(g(\nu)) \) is open in \( x \).

But, \( f \circ g(\nu) = (f \circ g \circ f^{-1})(\nu) \)

\( = (g \circ f \circ f^{-1})(\nu) \) - open in \( x \).

\( \Rightarrow \text{gof} \) is continuous.

d). We have

\( f \circ A = f \circ \text{gof} : A \rightarrow x \circ y \)

Since \( f \) and \( j \) are continuous, then \( f \circ j \) is continuous.

\( \Rightarrow f \circ A \) is continuous.

e). Let \( f : x \rightarrow y \) be continuous.

Let \( g : y \rightarrow z \) be such that \( g \circ x = f \).

Let \( \nu \) be open in \( z \).

\( \Rightarrow \nu = U \cap Z \) for some \( U \) open in \( y \)

\( g(\nu) = g(U) \cap Z = f(U) \)

\( \Rightarrow x \in f(U) \) and \( x \in x \)

\( \Rightarrow x \in f(U) \cap x \in f(U) \)

But \( f(U) \) is open in \( x \).

\( \Rightarrow g(U) \) is open in \( x \).

\( \Rightarrow g \) is continuous.

f). Let \( \text{hof}: x \rightarrow z \) where \( y \subseteq z \) be such that \( g \circ x = f \).

\( \Rightarrow h \circ f \) is continuous.

\( \Rightarrow h \circ \text{gof} \) is continuous.

\( \Rightarrow h \circ g \) is continuous.
Theorem. The pasting lemma

Let \( X = A \cup B \), where \( A \) and \( B \) are closed in \( X \). Let \( f: A \to Y \) and \( g: B \to Y \) be continuous. If \( f(x) = g(x) \) for every \( x \in A \cap B \), then \( f \) and \( g \) combine to give a continuous function \( h: X \to Y \), defined by setting

\[ h(x) = \begin{cases} f(x) & \text{if } x \in A \\text{ and } h(x) = g(x), \text{ if } x \in B \end{cases} \]

Proof. Let \( C \) be closed in \( Y \).

Then \( \bar{h}(C) = \bar{f}(C) \cup \bar{g}(C) \)

Since \( f \) and \( g \) are continuous,

\( \bar{f}(C) \) is closed in \( A \) and \( \bar{g}(C) \) is closed in \( B \).

But both \( A \) and \( B \) are closed in \( X \).

\( \bar{f}(C) \) and \( \bar{g}(C) \) are closed in \( X \).

Their union is also closed in \( X \).

\( \bar{h}(C) \) is closed in \( X \).

\[ h \] is continuous.

Note: In pasting lemma, \( A \) and \( B \) can be open.

Examples

1. Let \( h(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ 2x-1, & 1 \end{cases} \)

2. Let \( h(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ 2x-1, & x \in [1,2] \end{cases} \)

\( f \) and \( g \) are continuous.

and \( [0,1], [1,2] \) are closed.

\( [0,1] \cap [1,2] = \{1\} \)

\( h \) is continuous.

By pasting lemma, \( h \) is continuous.
Both $x$ and $x+1$ are continuous.

As $\cap_{1}^{5} A \cap_{1}^{3} J = \emptyset$

By pasting Lemma, $h$ is continuous.

$2)$ Let $h(x) = \begin{cases} x, & \text{if } x \in \mathbb{C}_{0,1} \\ x+1, & \text{if } x \in \mathbb{C}_{1,2} \end{cases}$

and

$f(1) = 1 + f(0) = f(2) = g(1)$

By pasting lemma,

$h$ is not continuous.

$3)$ Let $h(x) = \begin{cases} x, & \text{if } x \in \mathbb{C}_{0,1} \\ x+1, & \text{if } x \in \mathbb{C}_{1,2} \end{cases}$

Theorem: — Maps into products: —

Let $f: A \rightarrow X \times Y$ be given by the equation

$f(x) = (f_{1}(x), f_{2}(x))$

Then $f$ is continuous if the functions $f_{1}: A \rightarrow X$ and $f_{2}: A \rightarrow Y$

are continuous.

Note: The maps $f_{1}$ and $f_{2}$ are called the coordinate functions of $f$.

Proof: — Let $f_{1}: X \times Y \rightarrow X$ and $f_{2}: X \times Y \rightarrow Y$ be projecting maps.

Let $V$ be open in $X$.

$\Rightarrow f_{1}(V)$ is open in $X$ (as $\gamma$ is open in $Y$)

Similarly if $V$ is open in $Y$ then $f_{2}(V)$ is open in $Y$.

$x, y$ open in $X$
8. $T_2$ is Continuous

Note that $f_1 = T_1 \circ f$ and $f_2 = T_2 \circ f$.

Suppose that $f$ is continuous.

Since composition of continuous maps is continuous,

$f_1$ and $f_2$ are continuous.

Conversely, suppose that $f_1$ and $f_2$ are continuous.

Let $U \times V$ be open in $X \times Y$.

Then $U$ is open in $X$ and $V$ is open in $Y$.

Consider $f(U \times V) = f_1(U) \cap f_2(V)$.

Since $f_1$ and $f_2$ are continuous,

$f(U \times V)$ is open in $A$.

Then $A$ is continuous.

The Product Topology:

Defn.: Let $J$ be an index set. Given a set $X$, we define a $J$-tuple of elements of $X$ to be a function $X: J \rightarrow X$. If $\alpha$ is an element of $J$, we often denote the value of $X$ at $\alpha$ by $X(\alpha)$ rather than $X(\alpha)$. We call it the $\alpha$th coordinate of $X$. And we often denote the function $X$ itself by the symbol $\langle X(\alpha) : \alpha \in J \rangle$.

Let $J$ be an index set, $\{X_\alpha\}_{\alpha \in J}$ be collection of topological spaces.

The Cartesian product of this indexed family denoted by $\prod_{\alpha \in J} X_\alpha$ is defined to be the set of all $J$-tuples $\langle x_\alpha : \alpha \in J \rangle$ such that $x_\alpha \in X_\alpha$ for each $\alpha \in J$. 
Definition: Let \( \{ X_i \} \) be an indexed family of topological spaces. Let \( U \) take as a basis for a topology on the product space \( \prod_{i \in I} X_i \) the collection of all sets of the form \( \prod_{i \in I} U_i \), where \( U_i \in \mathcal{B}_i \) for each \( i \in I \).\( \bigcap \) is open in \( X_i \). The topology generated by this basis is called the box topology.

* Projection mapping: For \( \beta \in I \), \( \pi_\beta : \prod_{i \in I} X_i \rightarrow X_\beta \) is defined by \( \pi_\beta(x) = x_\beta \), where \( x_\beta \) is \( \beta \)-th coordinate of \( x = (x_i)_{i \in I} \).

* Defn: Let \( S \) denote the collection
  \[ S_\beta = \left\{ \pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta \right\} \]
  and
  \[ S = \bigcup_{\beta \in I} S_\beta \]
  Then \( S \) is a subbasis.

  The basis generated by \( S \) is the collection of finite intersections of elements of \( S \).

  The topology generated by this basis \( S \) is called the product topology on \( \prod_{i \in I} X_i \).

**Comparison of the box and product topologies:**

The box topology on \( \prod_{i \in I} X_i \) has as basis all sets of the form \( \prod_{i \in I} U_i \), where \( U_i \) is open in \( X_i \) for each \( i \). The product topology on \( \prod_{i \in I} X_i \) has as basis all sets of the form \( \prod_{i \in I} U_i \), where \( U_i \) is open in \( X_i \) for each \( i \) and \( U_\beta \) equals \( X_\beta \) except for finitely many values of \( \beta \).

Proof: By definition, the basis elements of box topo are of the type \( \prod_{i \in I} U_i \) where \( U_i \) is open in \( X_i \).

The product topo is generated by the subbasis

\[ S = \bigcup_{\beta \in I} S_\beta \]

\[ S_\beta = \left\{ \pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta \right\} \]
Let $\mathcal{B}$ be the basis for product topology. Let $b \in \mathcal{B}$. Then
\[ b = \bigotimes_{i \in J} U_i, \]
where $U_i \in \mathcal{S}_i$ for $i = 0, 1, \ldots, k$. If $i \neq j$, then $U_i \cap U_j = \emptyset$. 
Let $b_j \in \mathcal{B}_j$ for $j = 1, 2, \ldots, k$. 

We may assume that no two $b_j$'s belong to some $\mathcal{S}_j$. 

Let $\mathcal{B} = \bigotimes_{i \in J} U_i$. 

Then $b = \prod_{i \in J} (U_i \cap (\prod_{j \in J \setminus \{i\}} U_j))$. 

Here $U_i \times \prod_{j \in J \setminus \{i\}} U_j = \{x \in U_i \times \prod_{j \in J \setminus \{i\}} U_j : x_{i,j} \in U_i \}$ for each $i = 1, 2, \ldots, k$, and $x_{i,j}$ with $x_{i,j} \in U_i \times \prod_{j \in J \setminus \{i\}} U_j$. 

Let $x \in b \rightarrow x \in U_i \times \prod_{j \in J \setminus \{i\}} U_j$ for some each $i = 1, 2, \ldots, k$. 

$x_{i,i} \in U_i$ for $i = 1, 2, \ldots, k$. The remaining coordinates of $x$ belong to $x_{i,j}$'s. 

$b = \prod_{i \in J} U_i$ where $x_i = x_{i,i}$ if $i \not\in \{1, 2, \ldots, k\}$. 

Note: It is clear that every basis element for product topology is the basis element for box topology.

i) Box topology is finer than product topology.

ii) For finite products both bases are same.

Hence product topology = box topology.

iii) Arbitrary product of Hausdorff spaces is Hausdorff in both topologies.

Proof: Let $X_j$ be Hausdorff for each $j \in J$. Where $J$ is an indexed set.

Want: $\prod_{j \in J} X_j$ is Hausdorff.

Let $x, \bar{x} \in \prod_{j \in J} X_j$ such that $x \neq \bar{x}$.
Let $V = \prod_{p \in J} V_p$ and $s_p = V_p = x_p$ if $p \in J$. Then $s_p = x_p$, and $S_x = V_x$.

Obviously $V \neq \emptyset$, $V \neq E$.

$V$ and $V$ are open in box and product topologies.

Suppose $U \neq \emptyset$.

Let $z \in \bigcup_{p \in J} U_p$.

$z \in V$ and $z \in E$

$\Rightarrow z \in W_x = U_x$ and $z \in S_x = V_x$

$\Rightarrow z \in U \cap V$

$\Rightarrow$ contradiction.

$\Rightarrow \bigcup_{p \in J} U_p = \emptyset$

$\Rightarrow \Pi_{p \in J} U_p = \emptyset$

$\Rightarrow$ Hausdorff.

Theorem: Let $f: \Pi_{p \in J} X_p \rightarrow x_p$ be given by the equation $f(c) = (f(p))(c_p)$, where $f_p: A \rightarrow X_p$ for each $p$. Let $\Pi_{p \in J} X_p$ have the product topology. Then the function $f$ is continuous iff each function $f_p$ is continuous.

Proof: Let $f: \Pi_{p \in J} X_p \rightarrow x_p$ such that $f_p(\Pi_{p \in J} X_p) = x_p$.

Let $U_p$ be open in $x_p$.

$\Pi_{p \in J} (U_p)$ is a subbasis element for product topology.

$\Rightarrow U_p$ is open in $\Pi_{p \in J} X_p$.

$\Rightarrow$ $f_p$ is continuous.
Now, \[ f_\beta = \pi_\beta \circ f \]
\[ (\pi_\beta \circ f)(a) = \pi_\beta(f(a)) = \pi_\beta(f(a)) \subset c J \]
\[ = f_\beta(a) \neq a \]

If \( f \) is continuous,

then \( \pi_\beta \circ f \) is continuous.

(\( \pi_\beta \) is continuous and composition of continuous maps is continuous)

\( \Rightarrow f_\beta \) is continuous if \( \beta \in J \)

Conversely, suppose that \( f_\beta \) is continuous for each \( \beta \in J \).

To show that \( f \) is continuous.

Let \( U \) be a subbasis element of \( \pi X \).

\[ U = \pi_\beta^{-1}(U_\beta) \text{ for } \beta, \text{ where } U_\beta \text{ is open in } X_\beta \]

Now
\[ f_\beta(U) = f_\beta(\pi_\beta^{-1}(U_\beta)) = f \circ \pi_\beta^{-1}(U_\beta) \]
\[ = (\pi_\beta \circ f)_\beta(U_\beta) \]
\[ = f_\beta(U) \quad (\because f_\beta = \pi_\beta \circ f) \]

Since \( f_\beta : A \rightarrow x_\beta \) is continuous,

\[ f_\beta(U_\beta) \text{ is open in } A \]
\[ \Rightarrow f_\beta(U) \text{ is open in } A \]
\[ \Rightarrow f \text{ is continuous.} \]

**Note:** In the above theorem, the first part is true for the box topology, but converse is not true.

**Counter Example:**

Let \( \mathbb{R} = \prod_{i \in \mathbb{Z}} \]

Let $f : \mathbb{R} \rightarrow \mathbb{R}^n$ be defined by

$$f(t) = (\cos t, \sin t, t, t^2)$$

Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f_n(t) = t$$

Clearly each $f_n$ is continuous.

Claim: $f$ is not continuous.

Proof by contradiction.

Let $B = (-1, 1) \times (-1, 1) \times (0, 1) x \times (-1, 1) \mathbb{x} \cdots$

$B$ is open in $\mathbb{R}^n$.

Assume that $f(B)$ is open in $\mathbb{R}$.

Since $0 \in (-1/n, 1/n)$, $\delta = (0, 0, 0, \cdots) = f(0) \in B$.

Then $0 \in f(B)$ is open in $\mathbb{R}$.

There exists $\delta > 0$ such that $(-\delta, \delta) \in f(B)$.

Now, let $\pi_n$ be the $n$th projection for each $n$.

Applying $\pi_n$ on both sides we get

$$\pi_n(f(-\delta, \delta)) \subseteq \pi_n(B) = \pi_n\left(\pi_n(\mathbb{R}^n) \cap \left(-\frac{1}{n}, \frac{1}{n}\right) \times \cdots\right)$$

$$\Rightarrow (-\delta, \delta) \subseteq \left(-\frac{1}{n}, \frac{1}{n}\right)$$

for each

$$\Rightarrow -\frac{1}{n} \leq -\delta \leq \delta \leq \frac{1}{n}$$

which is contradicting to the fact that $\exists \delta > 0$ such that $\frac{1}{n} < \delta$.

$\therefore f(B)$ is not open in $\mathbb{R}$.

$\Rightarrow f$ is not continuous.
The Metric Topology:

Definition: A metric on a set \( X \) is a function
\[
d: X \times X \rightarrow \mathbb{R}
\]
having the following properties.
1. \( d(x,y) \geq 0 \), \( \forall x,y \in X \) - equality holds iff \( x=y \)
2. \( d(x,y) = d(y,x) \), \( \forall x,y \in X \)
3. (Triangle inequality) \( d(z,z) \leq d(x,z) + d(z,y) \), \( \forall x,y,z \in X \)

Given a metric \( d \) on \( X \), the number \( d(x,y) \) is often called the distance between \( x \) and \( y \) in the metric \( d \).

Given \( X \) and a metric \( d \), consider the set
\[ B(x,r) = \{ y \mid d(x,y) < r \} \]

\( B(x,r) \) is called as open ball centered at \( x \) and radius \( r \).

Let \( \mathcal{B} \) be the collection of such open balls. Then \( \mathcal{B} \) is a basis.

The topology generated by \( \mathcal{B} \) is called the topology induced by \( d \) or metric topology.

Thus \( U \) is open in \( X \) iff for every \( x \in U \) there is a ball \( B(x,r) \) such that \( B(x,r) \subseteq U \).

Definition: Let \( X \) be a topological space with topology \( T \). We say that \( X \) is a metrizable space if there is a metric \( d \) on \( X \) such that the topology induced by \( d \) is \( T \).

Examples:

1. Let \( X = \mathbb{R} \), \( d(x,y) = |x-y| \), \( \forall x,y \in \mathbb{R} \). Then \( d \) is a metric on \( \mathbb{R} \).

\[
B(x,r) = \{ y \mid d(x,y) < r \} = \{ y \mid |x-y| < r \} = \{ y \mid y-x < r \text{ and } y-x > -r \} = (x-r,x+r)
\]

The metric topology on \( \mathbb{R} \) is the standard topology on \( \mathbb{R} \).
2) Let $X = IR$, $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$.

They $d$ is discrete metric on $IR$.

Now,

$B(x, r) = \{ y \mid d(x,y) < r \}$

If $r \geq 1$ then $d(x,y) = 0$

$\Rightarrow B(x,r) = IR$, if $r \geq 1$

If $r < 1$, then $d(x,y) < 1$

$\Rightarrow d(x,y) = 0 \Rightarrow x = y$

$\Rightarrow B(x,r) = \{ x \}$, if $r < 1$

= one point sets are open in $IR$.

We know every subset is union of one point sets and arbitrary union of open sets is open.

= Each subset is open in $IR$.

The topology induced by $d$ is discrete topology.

• Note: $d, d'$-metrics on $X$ induces topology $\tau, \tau'$ respectively, then $\tau'$ is finer than $\tau$ if for every $x \in X$ and $r > 0$, exists $r' > 0$ such that $B_r(x, r') \subseteq B_r(x, r)$.

• Definition: Let $X$ be a metric space with metric $d$. A subset $A$ of $X$ is said to be bounded if there is some number $M$ such that $d(a_1, a_2) \leq M$ for every pair $a_1, a_2$ of points of $A$. If $A$ is bounded and nonempty, the diameter of $A$ is defined to be the number $\text{diam} \ A = \sup \{ d(a_1, a_2) \mid a_1, a_2 \in A \}$.

• Theorem: Let $X$ be a metric space with metric $d$. Define $\overline{d} : XX \rightarrow IR$ by the equation $\overline{d}(x,y) = \min \{d(x,y), 1\}$.

Then $\overline{d}$ is a metric that induces the same topology as $d$. 

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The metric $d$ is called the Standard bounded metric corresponding to $d$.

Example 5: If $f : X \to Y$ is a continuous function on one metrizable space to another metrizable space and $A$ is bounded in $X$.

Check whether $f(A)$ is bounded in $Y$ or not.

Solution: The above statement is not true.

Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^n$.

Define $f : X \to Y$ by $f(x) = x$, $\forall x \in X$.

Which is continuous map.

Since every set in $X$ is open as well as closed.

For $A = \mathbb{C} \cdot \mathbb{N}$, then

$d(A) = 1 \leq 2 = M$.

$\Rightarrow$ $A$ is bounded in $X$.

but $f(A) = \mathbb{C} \cdot \mathbb{N}$.

Then $d(f(A)) = 2n$ which is unbounded.

$\Rightarrow$ $f(A)$ is not bounded in $Y$.

Boundedness is not a topological property.

- Metrics on $\mathbb{R}^n$:

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

1) Define $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d(x, y) = \|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}.$$ 

Then $d$ is metric on $\mathbb{R}^n$. It is called an Euclidean metric on $\mathbb{R}^n$.

2) Define $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$g(x, y) = \max \{ |x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n| \}.$$ 

Then $g$ is metric on $\mathbb{R}^n$. It is called a Square metric on $\mathbb{R}^n$. 

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Theorem: The topologies on $\mathbb{R}^n$ induced by the euclidean metric $d$ and the square metric $\gamma$ are the same as the product topology on $\mathbb{R}^n$. 

Proof: To prove that $d$ and $\gamma$ induce the same topology.

First we prove that, for $x, y \in \mathbb{R}^n$,

$$\gamma(x, y) = d(x, y) \leq \sum_{i=1}^{n} \gamma(x_i, y_i) \leq \sqrt{n} \gamma(x, y)$$

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$

$$\gamma(x, y) = \max \{ |x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n| \}$$

$$= |x_i - y_i| \text{ for some } i$$

Now

$$|x_i - y_i| = \sqrt{|x_i - y_i|^2} = \sqrt{(x_i - y_i)^2}$$

$$\therefore \gamma(x, y) = \sqrt{(x_i - y_i)^2} \leq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

$$= d(x, y)$$

$$\gamma(x, y) \leq d(x, y) \quad (1)$$

Since $|x_k - y_k| \leq |x_i - y_i|$, $\forall k$,

$$\therefore (x_k - y_k)^2 \leq (x_i - y_i)^2, \forall k = 1, 2, \ldots, n$$

$$\Rightarrow \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \leq \sqrt{n} \sqrt{(x_i - y_i)^2}$$

$$\Rightarrow d(x, y) \leq \sqrt{n} \gamma(x, y) \quad (2)$$

By equations (1) and (2) we get

$$\gamma(x, y) \leq d(x, y) \leq \sqrt{n} \gamma(x, y)$$
Let \( \gamma, \gamma' \) be topologies induced by \( d \) and \( s \) respectively.

Want: \( \gamma' \subseteq \gamma \)

Let \( x \in \mathbb{R}^n \), \( B_{\delta}(x, r) \) be basis element of \( \gamma' \)

Consider \( B_{\delta}(x, r) \),

obviously \( x \in B_{\delta}(x, r) \)

Let \( y \in B_{\delta}(x, r) \Rightarrow d(x, y) < \delta \)

\[ \delta (x, y) < \delta \]

\[ y \in B_{\delta}(x, r) \]

\[ B_{\delta}(x, r) \subseteq B_{\delta}(x, r) \]

\[ \gamma' \subseteq \gamma \] \( \text{(3)} \)

Let \( B_{\delta}(x, r) \) be a basis element of \( \gamma \)

Want \( \epsilon > 0 \) such that \( B_{\delta}(x, \epsilon) \subseteq B_{\delta}(x, \epsilon) \)

We want \( \epsilon > 0 \) such that \( \delta(x, y) < \epsilon \Rightarrow d(x, y) < \delta \)

Take \( \epsilon = \frac{\epsilon}{\sqrt{n}} \)

\[ d(x, y) \leq \frac{\epsilon}{\sqrt{n}} \]

\[ \frac{\epsilon}{\sqrt{n}} \leq \frac{\epsilon}{\sqrt{n}} \]

\[ \Rightarrow d(x, y) < \epsilon \]

\[ \gamma \subseteq \gamma' \] \( \text{(4)} \)

By equations \( \text{(3)} \) and \( \text{(4)} \) we get

\( \gamma = \gamma' \)

To show that \( \gamma' \) is the product topology on \( \mathbb{R}^n \).

Let \( B = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \) be a basis element of the product topology containing

\[ x = (x_1, x_2, \ldots, x_n) \]

\[ x_i \in (a_i, b_i), \quad 2 \epsilon = (a_2, b_2) \quad \cdots \quad 2 \epsilon \in (a_n, b_n) \]

\[ \exists \epsilon_i \text{ such that } (x_i - \epsilon_i, x_i + \epsilon_i) \subseteq (a_i, b_i) \]

\[ \forall i = 1, 2, \ldots, n \]

Let \( \alpha = \min \{ \epsilon_1, \epsilon_2, \ldots, \epsilon_n \} \)

Consider \( B_{\alpha}(x, \alpha) \)
Let \( y \in B_\epsilon(x, r) \)

\[
g(x, y) = \max \{ |x_i - y_i|, |x_2 - y_2|, \ldots, |x_n - y_n| \} < \epsilon
\]

\[
\Rightarrow |x_i - y_i| < \epsilon, \quad \forall i \in \{1, 2, \ldots, n\}
\]

\[
\Rightarrow |x_i - y_i| < \epsilon_i, \quad \forall i
\]

\[
\Rightarrow y_i \in (x_i - \epsilon_i, x_i + \epsilon_i), \quad \forall i
\]

\[
y = (y_1, y_2, \ldots, y_n) \in (x_1 - \epsilon_1, x_1 + \epsilon_1) \times (x_2 - \epsilon_2, x_2 + \epsilon_2) \times \cdots \times (x_n - \epsilon_n, x_n + \epsilon_n) \subseteq B
\]

Thus if \( y \in B_\epsilon(x, r) \) then \( y \in B \)

\[
B_\epsilon(x, r) \subseteq B
\]

i.e. \( \gamma' \) is finer than product topology.

Conversely suppose that \( B_\epsilon(x, r) \in \gamma' \)

Let \( x = (x_1, x_2, \ldots, x_n) \)

Let \( B = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \)

Then \( B \) is a base basis element of product topology

Let \( y = (y_1, y_2, \ldots, y_n) \in B \)

\[
y_i \in (x_i - \epsilon, x_i + \epsilon)
\]

\[
\Rightarrow |x_i - y_i| < \epsilon, \quad \forall i
\]

\[
\Rightarrow \max |x_i - y_i| < \epsilon, \quad \forall i
\]

\[
g(x, y) < \epsilon
\]

\[
y \in B_\epsilon(x, r)
\]

\[
\Rightarrow \gamma' \text{ product topology}
\]

From equations 5 and 6 we get

\[
\gamma' \text{ product topology}
\]

Hence \( \gamma = \gamma' = \text{product topology on } \mathbb{R}^n \).
Ex. Show that the Euclidean metric $d$ on $\mathbb{R}^n$, as follows.

If $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$cx = (cx_1, cx_2, \ldots, cx_n)$$

$$x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

a). Show that $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

b). Show that $|x \cdot y| \leq \|x\| \cdot \|y\|$.

c). Show that $\|x + y\| \leq \|x\| + \|y\|$.

d). Verify that $d$ is a metric.

Solution:

a). $(x \cdot y) + (x \cdot z) = \sum_{i=1}^{n} x_iy_i + \sum_{i=1}^{n} x_iz_i$

$$= \sum_{i=1}^{n} x_i(y_i + z_i)$$

$$= x \cdot (y + z).$$

b). We know that $0 \leq \|x - ty\|^2$

$$= |x - ty| |x - ty|$$

$$= x \cdot x - tx \cdot y - tyx + t^2y \cdot y$$

$$= \|x\|^2 - 2txy + t^2\|y\|^2$$

Assume that $y \neq 0$

Let $t = \frac{x \cdot y}{\|y\|^2}$

$$0 \leq \|x\|^2 - 2xy \cdot x \cdot y + \left(\frac{x \cdot y}{\|y\|^2}\right)^2 \|y\|^2$$

$$= \|x\|^2 - 2\left(\frac{x \cdot y}{\|y\|^2}\right)^2 \|y\|^2 + \left(\frac{x \cdot y}{\|y\|^2}\right)^2 \|y\|^2$$

$$= \|x\|^2 - \left(\frac{x \cdot y}{\|y\|^2}\right)^2 \|y\|^2$$

$$\Rightarrow (xy)^2 \leq \|x\|^2 \cdot \|y\|^2$$
Taking square root on both sides we get

\[(x-y) = \|x\| \cdot \|y\|\]

(c) \[\|x+y\|^2 = \left(\sqrt{(x+y)(2x+y)}\right)^2 = (x+y)^2\]

\[\sum_{i=1}^{n} \left(x_i^2 + 2x_iy_i + y_i^2\right) = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} 2x_iy_i + \sum_{i=1}^{n} y_i^2\]

\[\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2\]

\[= \|x\|^2 + 2\|x\|\|y\| + \|y\|^2\]

Taking square root on both sides we get

\[\|x+y\| \leq \|x\| + \|y\|\]

d) To show that \(d(x,y) = \|x-y\|\) is a metric

\[\text{since } \|x-y\| = 0 \implies d(x,y) = 0\]

and

\[d(x,y) = 0 \iff \|x-y\| = 0 \iff \sum_{i=1}^{n} \frac{1}{2}(x_i-y_i)^2 = 0\]

\[\iff x_i = y_i, \forall i\]

\[\implies d(x,y) = 0 \iff x = y\]

ii) \[d(x,y) = \|x-y\| = \sqrt{\sum_{i=1}^{n} \frac{1}{2}(x_i-y_i)^2}\]

\[= \sqrt{\sum_{i=1}^{n} \frac{1}{2}(x_i-y_i)^2}\]

\[= \|y-x\| = d(y,x)\]

\[\therefore d(x,y) = d(y,x)\]
\[ d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| \]

\[ = d(x, y) + d(y, z) \]

\[ \therefore d(x, z) \leq d(x, y) + d(y, z) \]

\[ \therefore d(x, y) = \|x - y\| \text{ is metric.} \]

- **Uniform Topology:**
  Given an index set \( J \) and given points \( x = (x_k)_{k \in J} \) and \( y = (y_k)_{k \in J} \) of \( IR^J \).

Let us define a metric \( \bar{g} \) on \( IR^J \) by the equation

\[ \bar{g}(x, y) = \sup \left\{ \frac{d(x_k, y_k)}{k \in J} \right\} \]

Where \( d \) is the standard bounded metric on \( IR \).

\[ \text{ie } d(x_k, y_k) = \min \left\{ |x_k - y_k|, 1 \right\} \]

\( \bar{g} \) is called as uniform metric on \( IR^J \). The topology induced by \( \bar{g} \) is called uniform topology on \( IR^J \).

- **Theorem:** The uniform topology on \( IR^J \) is finer than the product topology and coarser than the box topology.

- **Proof:** Let \( z = (z_k)_{k \in J} \in IR^J \) and let \( B = \bigcup_{k \in J} U_k \) be a basis element of product topology.

Let \( x_1, x_2, \ldots, x_n \in J \) such that \( U_k \neq IR \) and \( U_k = IR \) if \( x_k \neq x_i \) for any \( i \).

Since \( U_k \) are open in \( IR \) and \( d \) induces the standard topology on \( IR \), \( \bar{g} \) \( \in \tau \) such that

\[ B_{\bar{g}}(x_k, e_i) \subseteq U_k. \]

Let \( \epsilon = \min \{ e_1, e_2, \ldots, e_n \} \)

Let \( B' = B_{\bar{g}}(x, \epsilon) \)

Let \( y \in B' \Rightarrow \bar{g}(x, y) < \epsilon \)

\[ \Rightarrow \sup \left\{ d(x_k, y_k) \right\} < \epsilon \]

\[ \Rightarrow d(x_k, y_k) < \epsilon \leq \epsilon_i, \quad \forall \ x_k \in J \]
\[ y_k \in B_\delta(x_k, e) \subset U_k \quad \text{and} \quad y_k \in U_k = 1 \quad \text{for any } k \]

\[ y = (y_k) \in \prod_{k \in J} U_k = B \]

Thus, \( y = (y_k) \in \prod_{k \in J} U_k = B \)

\[ \Rightarrow \text{Product topology } \subset \text{Uniform topology.} \]

To prove that Uniform topology \( \subset \) Box topology.

Let \( x = (x_k)_{k \in J} \) and consider \( B \subset (x, e) \)

Let \( U_k = (x_k - e/2, x_k + e/2) \subset 1 \)

Each \( U_k \) is open in 1 \( \mathbb{R} \)

Let \( B = \prod_{k \in J} U_k = B \) is a basic element for box topology.

since, \( x_k \in U_k \), \( \forall k \in J \)

\[ x = (x_k)_{k \in J} = \prod_{k \in J} U_k = B \]

Let \( y \in B \Rightarrow y_k \in U_k \)

\[ \Rightarrow |x_k - y_k| < e/2, \quad \forall k \in J \]

\[ \Rightarrow \delta(x_k, y_k) \leq |x_k - y_k| < e/2, \quad \forall k \in J \]

\[ \Rightarrow \sup_k \delta(x_k, y_k) \leq e/2 \]

\[ \Rightarrow \delta(x, y) \leq e/2 < e \]

\[ \Rightarrow y \in B \subset (x, e) \]

\[ \Rightarrow \text{Uniform topology } \subset \text{Box topology} \]

Theorem: Let \( f : x \to y \), let \( x \) and \( y \) be metrizable with metrics \( d_x \) and \( d_y \) respectively. The continuity of \( f \) is equivalent to the requirement that given \( x \in X \) and given \( e > 0 \), there exists \( d > 0 \) such that

\[ d_x(x, y) < d \Rightarrow d_y(f(x), f(y)) < e \]
The Sequence Lemma: Let \( X \) be a topological space.

Let \( A \subseteq X \). If there is a sequence of points of \( A \) converging to \( x \), then \( x \in \overline{A} \), the converse holds if \( X \) is metrizable.

Proof: Suppose that \( x_n \to x \). Let \( U \) be any nbhd of \( x \). Then \( U \) contains many terms of \( \{x_n\} \).

Hence, \( U \cap A \neq \emptyset \) (\( \because x \in \overline{A} \), \( \forall n \))

\( \Rightarrow x \in \overline{A} \)

Conversely, assume that \( x \in \overline{A} \) and \( X \) is metrizable.

Want a sequence \( \{x_n\} \subseteq A \) such that \( x_n \to x \).

The open ball \( B(x, \frac{1}{n}) \) of \( x \) intersects \( A \).

Let \( x_n \in B(x, \frac{1}{n}) \cap A, \forall n \)

Thus \( \{x_n\} \subseteq A \)

To show that \( x_n \to x \).

Let \( U \) be a nbhd of \( x \) in \( X \).

As \( U \) is open \( \exists \epsilon > 0 \) such that \( B(x, \epsilon) \subseteq U \).

Let \( N \) be a positive integer such that \( \frac{1}{N} < \epsilon \).

Consider \( B(x, \frac{1}{n}) \) for \( n > N \).

Let \( y \in B(x, \frac{1}{n}) \Rightarrow d_X(x, y) < \frac{1}{n} \leq \frac{1}{N} < \epsilon \)

\( \Rightarrow y \in B(x, \epsilon) \subseteq U \)

\( \Rightarrow B(x, \frac{1}{n}) \subseteq U, \forall n \geq N \)

\( \Rightarrow x_n \in U, \forall n \geq N \)

\( \Rightarrow x_n \to x \) (\( \because x_n \in B(x, \frac{1}{n}) \))

Theorem: Let \( f: X \to Y \). If the function \( f \) is continuous, then for every convergent sequence \( x_n \to x \) in \( X \), the sequence \( f(x_n) \) converges to \( f(x) \). The converse holds if \( X \) is metrizable.

Proof: Suppose that \( f \) is continuous.

Suppose \( x_n \to x \) in \( X \).

To show that \( f(x_n) \to f(x) \) in \( Y \).

Let \( U \) be a nbhd of \( f(x) \) in \( Y \).

\( \Rightarrow x \in f^{-1}(U) \)
Since \( f \) is continuous
\[
\Rightarrow f(c) \text{ is open in } X
\]
as \( x_n \to x, \exists N \text{ such that } x_n \in f(U), \forall n \geq N
\]
\[
\Rightarrow f(x_n) \in U, \forall n \geq N
\]
\[
\Rightarrow f(x_n) \to f(x)
\]
Conversely suppose that \( x_n \to x \) in \( X \) and
\[
f(x_n) \to f(x) \text{ in } Y, \text{ and } X \text{ is metrizable.}
\]
To show that \( f \) is continuous.
Let \( A \subseteq X \). To show that \( f(A) \subseteq f(A) \)
let \( f(x) \in f(A) \) for some \( x \in A \)
by sequence lemma, \( \exists x_n \in A \) such that \( x_n \to x \)
\[
\Rightarrow f(x_n) \to f(x) \quad \text{(by hypothesis)}
\]
\[
\Rightarrow f(x) \in f(A) \quad \text{(*)}
\]
\[
\Rightarrow f(A) \subseteq f(A)
\]
\[
\Rightarrow f \text{ is continuous.}
\]

- Uniform Continuity :-

Defn. - Let \( f_n : X \to Y \) be a sequence of functions
from the set \( X \) to the metric space \( Y \). Let \( d \) be
the metric for \( Y \). We say that the sequence
\((f_n)\) converges uniformly to the function \( f : X \to Y \) if
given \( \epsilon > 0 \), there exists an integer \( N \) such that
\[
d(f_n(x), f(x)) < \epsilon \quad \text{for all } n \geq N \text{ and all } x \in X.
\]

- Uniform Limit Theorem : - Let \( f_n : X \to Y \) be a seq-
quence of continuous functions from the topological space \( X \)
to the metric space \( Y \). If \((f_n)\) converges uniformly to \( f \)
then \( f \) is continuous.

Proof. We have \( f : X \to Y \) : \( Y \) is a metric space.
Claim: \( f \) is continuous.
Let \( V \) be open in \( Y \).
To show (that \( f(V) \) is open in \( X \).
Suppose \( f(V) = \emptyset \) then it is open in \( X \).
Suppose \( f(V) \neq \emptyset \)
Let \( x \in f^{-1}(V) \Rightarrow f(x) \in V \), as \( V \) is open.

1. \( B(f(x_0), \varepsilon) \subseteq V \) for some \( \varepsilon > 0 \).

Since \( f_n \to f \) uniformly, \( \exists N \) such that
\[
d(f_n(x), f(x)) < \varepsilon/3, \quad \forall n \geq N, \quad x \in X.
\]

In particular
\[
d(f_N(x), f(x)) < \varepsilon/3, \quad \forall x \in X.
\]

Since \( f_n \) is continuous and \( B(f_N(x_0), \varepsilon/3) \) open,
\[
\Rightarrow f_N^{-1}(B(f_N(x_0), \varepsilon/3)) \text{ is open in } X.
\]

As \( f_N(x_0) \in B(f_N(x_0), \varepsilon/3) \)
\[
\Rightarrow x_0 \in f_N^{-1}(B(f_N(x_0), \varepsilon/3)) \Rightarrow \exists \text{ open } U \text{ in } X \text{ such that } x_0 \in U \subseteq f_N^{-1}(B(f_N(x_0), \varepsilon/3)).
\]

\[
\Rightarrow f_N(U) \subseteq B(f_N(x_0), \varepsilon/3)
\]

If \( x \in U \Rightarrow x \in f_N(U) \Rightarrow f_N(x) \in f_N(U) \Rightarrow f(x) \in f(U) \Rightarrow f(U) \subseteq f(U).
\]

Hence \( x \in f(U) \Rightarrow d(f(x), f(x_0)) < \varepsilon/3 \)
\[
d(f(x), f(x_0)) \leq d(f(x), f(x_0)) + d(f(x_0), f(x_0)) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\]

Therefore \( f(x) \in f(U) \subseteq f(U) \Rightarrow \Rightarrow f(U) \) is open in \( X \).

\[ \Rightarrow f \text{ is continuous.} \]

\[ \begin{align*}
\text{Example:} & \quad \mathbb{R}^w \text{ in the box topology is not metrizable.} \\
\text{Solution:} & \quad \text{Let } A = \{ (a_1, a_2, \cdots) \mid a_i > 0 \} \cap \mathbb{R}^w \\
& \quad \text{to show that } \bar{a} = (0, 0, \cdots) \notin \bar{A}
\end{align*} \]
(IR^n is a countable product of IR with itself)

Let B = \{(a_i, b_i) \times (a_2, b_2) \times \ldots \} be an open set in IR^n containing \(0 \in (a_i, b_i), \forall i\)

Let \(b = (b_1, b_2, \ldots)\) since \(b_i \in (a_i, b_i), b \in B\)

Also \(\forall i, 0 < \frac{1}{2^n}, b_i \cap a_i \Rightarrow B \cap A \neq \phi\)

\[\Rightarrow a \in A\]

Assume that IR^n is metrizable in box topology, then by sequence lemma there is a sequence
\[\alpha_n = (a_{n1}, a_{n2}, \ldots, a_{nn}, \ldots)\] of terms of A such that \(\alpha_n \to a\)

Here all a_{ii}’s are positive

\[\Rightarrow a_{nn} > 0, \forall n\]

Let \(B = (-a_{11}, a_{11}) \times (-a_{22}, a_{22}) \times \ldots \times (-a_{nn}, a_{nn}) \times \ldots\)

Then B is open in IR^n and contain \(0 \to a\) as \(\alpha_n \to a, \forall n\) such that \(\alpha_n \in B, \forall n \in N\)

In particular \(\alpha_n \in B\)

\[\Rightarrow (a_{11}, a_{22}, \ldots, a_{nn}, \ldots) \cap (-a_{11}, a_{11}) \times (-a_{22}, a_{22}) \times \ldots \times (-a_{nn}, a_{nn}) \times \ldots\]

\[\Rightarrow a_{nn} \in (-a_{nn}, a_{nn}) \Rightarrow \text{Contradiction} \Rightarrow (\text{a_{nn} < a_{nn} < a_{nn}})

\[\Rightarrow \text{our assumption was wrong}\]

IR^n is not metrizable in box topology.

* The Quotient Topology:

Definition: Let X and Y be two topological spaces and \(p: X \to Y\) a surjective map. The map p is said to be a quotient map provided that U is open in Y iff \(p(U)\) is open in X.

This map is also called a strong continuity map or stronger continuity.
Ex. If $p: X \to Y$ is a surjective map. Then show that $p$ is quotient map iff closed subset $A$ of $Y$ gives $p^{-1}(A)$ is closed in $X$.

So, let $p: X \to Y$ be a surjective map and which is quotient map.

Let $A$ be any closed subset of $Y$.

$\Rightarrow p^{-1}(Y - A)$ is open in $X$.

but $p^{-1}(Y - A) = p^{-1}(Y) - p^{-1}(A) = X - p^{-1}(A)$

then $X - p^{-1}(A)$ is open in $X$.

$\Rightarrow p^{-1}(A)$ is closed in $X$.

B. Saturated set: If $p: X \to Y$ is a surjective map then a subset $C$ of $X$ is said to be a saturated set if $C = p^{-1}(V), \forall V \subseteq Y$.

Remark: By using properties of saturated set an equivalent definition of quotient map can be defined as $p: X \to Y$ is said to be a quotient map iff $p$ is continuous map and $p$ maps saturated open sets of $X$ to open sets of $Y$.

The above definition is true since $p$ is quotient map then $p$ is continuous and whenever $C = p^{-1}(U)$ open in $X$ (i.e. $C$ is saturated open set in $X$)

$\Rightarrow U$ is open in $Y$.

$p(C) = U$ is image of saturated open set is open in $Y$.

Quotient Topology: If $X$ is a topological space and $A$ is a set such that $p: X \to A$ is a quotient map then the topology defined on $A$ by using $p$ is called as quotient topology.

Ex. If $(X, T)$ is any topological space such that $p: X \to A$ is a quotient map then there is a topology
on $A$ by using $p$.

- **Solution:** Let $(x, \gamma)$ be any topological space.
  
  $p : X \rightarrow A$ be any quotient map.

  1) $\phi \in A \Rightarrow p^{-1}(\phi) = \phi \in T$ ($\gamma$ is topology on $X$)
  
  Also $A \subseteq A \Rightarrow p^{-1}(A) = X \in \gamma$ ($\gamma$ is topology on $X$)

  Hence both $\phi_{ym}$ and $A$ are open in $A$.

  3) Let $\{U_k\}_{k \in J}$ be any arbitrary collection of open sets in $A$.

  $\Rightarrow \{p^{-1}(U_k)\}_{k \in J}$ be collection of open sets in $X$.

  Since $(X, \gamma)$ is a topological space,

  $\Rightarrow U_k \cap \bigcup_{k \in J} U_k$ is open in $X$.

  $\Rightarrow U_k \in \gamma$ is open in $A$.

  $\Rightarrow \bigcup_{k \in J} p^{-1}(U_k) = p^{-1}\left(\bigcup_{k \in J} U_k\right)$ is open in $X$.

  ($p$ is surjective map)

  Which becomes saturated open sets in $X$.

  and $p$ is quotient map.

  $\Rightarrow U_k$ is open in $A$.

  3) Let $\{U_j\}_{j=1}^n$ be any finite collection of open sets in $A$

  $\Rightarrow \{p^{-1}(U_j)\}_{j=1}^n$ becomes a collection of open sets in $X$.

  ($\because (X, \gamma)$ is topological space)

  $\Rightarrow \bigcap_{j=1}^n p^{-1}(U_j)$ is open in $X$

  $\Rightarrow p^{-1}\left(\bigcap_{j=1}^n U_j\right)$ is open in $X$ ($p$ is onto)

  Which becomes saturated open sets in $X$ as $p$ is quotient map

  $\Rightarrow \bigcap_{j=1}^n U_j$ is open in $A$. 
A is topology on $X$.

Hence it is a quotient topology.

**Theorem:** Let $P: X \rightarrow Y$ be a quotient map. Let $Z$ be a space and let $g: X \rightarrow Z$ be a map that is constant on each set $P^{-1}(y)$, for $y \in Y$. Then $g$ induces a map $f: Y \rightarrow Z$ such that $P \circ g = g$. The induced map $f$ is continuous if $g$ is continuous. If $f$ is a quotient map iff $g$ is a quotient map.

**Proof:**

Step 1: Given that $P: X \rightarrow Y$ is a quotient map and $g: X \rightarrow Z$ is a constant map on each set $P^{-1}(y)$, $y \in Y$ which can be represented geometrically from \( \text{Fig. ca) } \).

\[ \begin{array}{ccc}
X & \xrightarrow{P} & Y \\
\downarrow g & & \downarrow f \\
Y & \xrightarrow{f} & Z \\
\end{array} \]

Define a function $f: Y \rightarrow Z$ such that $P \circ f = g$ with the property that $g: X \rightarrow Z$ is a constant on each set $P^{-1}(y)$, $y \in Y$ which can be represented as \( \text{Fig. (b))} \).

\[ \begin{array}{ccc}
X & \xrightarrow{P} & Y \\
\downarrow g & & \downarrow f \\
Y & \xrightarrow{f} & Z \\
\end{array} \]

As $g$ is constant map on each set of $P^{-1}(y)$, $y \in Y$

\[ g(p(y)) = g(P^{-1}(y)) \] which is one point in $Z$.

To construct $f: Y \rightarrow Z$ such that $f(P(x)) = g(x)$, $x \in X$ where $g$ is constant map.

\[ g \text{ is continuous.} \]

Hence the inverse image of open set is open under $g$.

Let $U$ be any open set in $Z$.

\[ g(U) = (P \circ f)(U) \text{ is open in } X \]

\[ g(U) = P^{-1}(f(U)) \text{ - becomes saturated open sets in } X \]

and $P$ is quotient map.

\[ f(U) \text{ is open in } Y \]

\[ f \text{ is continuous.} \]
Conversely if $f$ is continuous and $f \circ p$ is also continuous, since $p$ is quotient map
\[ f \circ p = g \circ p = g \] is continuous.

**Step II:** To show that $f$ is quotient map if $g$ is quotient map.

\[ f(p(x)) = g(x), \ x \in X \]

Now if $f$ is quotient map
\[ f(p(x)) \text{ is a continuous map} \]

Hence $f$ is also continuous and surjective.

On the other hand as $f$ is quotient map, $f$ maps a saturated open subset of $Y$ to open subset in $Z$.

With the property that $f(p(x)) = g(x), \ x \in X$.

Hence $g$ also maps saturated open subset of $X$ to open subset in $Z$.

Hence $g$ is quotient map.

Let $f(p(x)) = g(x), \ x \in X$ such that $g$ is quotient map.

To show that $f$ is quotient map.

Since $g$ is quotient map
\[ g \text{ is surjective hence from above equation } f \text{ is also surjective.} \]

It remains to show that $V$ is open in $Z$.

\[ f^{-1}(V) \text{ is open in } Y \]

Let $U$ be open in $Z$.

\[ g^{-1}(U) \text{ is open in } X \]

\[ f^{-1}(g^{-1}(U)) \text{ is open in } Y \]

From eqn (1).

\[ f^{-1}(V) = f^{-1}(g^{-1}(U)), \forall V \subseteq Z \text{ which is open in } Y \]

Hence $f^{-1}(V)$ is open in $Y$.

\[ f \text{ is quotient map.} \]

---

*Retraction map:* If $X$ is any topological space and $A$ is any subset of $X$, then $r: x \rightarrow A$ is called a retraction map if $r$ is continuous and $r|a = a, A \subseteq A$.

*Ex.* Show that every retraction map is a quotient map.

*Solution:* Let $X$ be any topological space.

and $r: x \rightarrow A$ is a retraction map.
Define $f: A \to A$ by $f(a) = a$, $\forall a \in A$

Then $f$ is a well-defined map, which is an identity map and hence continuous, and also a quotient map.

with the property that

$$\forall a \in A, f(a) = \text{Id}(a)$$

where $\text{Id}$ is an identity map and hence a quotient map.

$\therefore \text{a}$ is a quotient map.