Countability and Separation Axioms

Definition: A space $X$ is said to have a Countable basis at $x$ if there is a Countable collection $\mathcal{B}$ of neighborhoods of $x$ such that each neighborhood of $x$ contains at least one of the elements of $\mathcal{B}$. i.e., if $V$ is any open set containing $x$, then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq V$.

If $X$ has Countable basis at every $x \in X$, then $X$ is called a first Countable Space.

Example: Metrizable space is first Countable.

Solution: Let $X$ be a metric space and let $x \in X$.

Let $\mathcal{B} = \{B_n = B(x, \frac{1}{n}) : n \in \mathbb{N}\}$

$\mathcal{B}$ is Countable Collection of Nbd.s of $x$.

Let $U$ be any open set containing $x$.

$\exists B(x, \frac{1}{r}) \subseteq U$ for some $r > 0$.

Then $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < r$.

$\Rightarrow B(x, \frac{1}{n}) \subseteq B(x, \frac{1}{r}) \subseteq U$

$\Rightarrow B$ is a Countable basis at $x$.

$\Rightarrow X$ has Countable basis at every $x \in X$.

$\Rightarrow X$ is first Countable.

Theorem: Let $X$ be a topological space.

a) Let $A$ be a subset of $X$. If there is a sequence of points of $A$ converging to $x$, then $x \in A$.

The converse holds if $X$ is first Countable.

b) Let $f : X \to Y$. If $f$ is Continuous, then for every convergent sequence $x_n \to x$ in $X$, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if $X$...
is first-countable.

- Definition: 
  1. A subset $A$ of a space $X$ is said to be dense in $X$ if $\overline{A} = X$.
  2. If a space $X$ has a countable basis for its topology, then $X$ is called second-countable.
  3. If every open covering of $X$ has a countable subcovering, then $X$ is called Lindelöf.

- Theorem: A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

- Proof: 
  Given that $X$ is second-countable space.
  - $X$ having countable basis $B$, and hence by property of subspace topology,
  - $B = \mathcal{B} \cap X$ becomes a basis for the subspace $A$ of $X$.
  - Which is countable basis.
  - $A$ is second-countable.
  - Subspace of second-countable space is second-countable.
  - To show that countable product of second-countable space is second-countable.
  - If $B_i$ is a countable basis for the space $X_i$.
  - then the collection of all products $\mathcal{T}_U^i$, where $U_i \in B_i$ for finitely many values of $i$, and $U_i \subseteq X_i$ for all other values of $i$, is a countable basis for
1. Countable Product of Second Countable Space is Second Countable.

The proof for the first countability is similar.

Example. i) Second Countable \( \Rightarrow \) First Countable

ii) Second Countable \( \Rightarrow \) Lindelof

iii) Second Countable \( \Rightarrow \) Separable

Solution. Let \( X \) be a second countable space.

Let \( \{B_n\} \) be a countable basis for \( X \).

1) Let \( x \in X \)

Let \( A = \{B_n : x \in B_n\} \)

\( \Rightarrow A \) is countable collection of nbds. of \( x \).

Let \( U \) be any nbhd. of \( x \); then

\( x \in B_{n_0} \subseteq U \), for some \( B_{n_0} \in A \).

\( B_{n_0} \in A \).

Thus \( A \) is a countable collection of nbds. of \( x \) such that every nbhd. of \( x \) contains a member of \( A \).

\( \Rightarrow A \) is countable basis at \( x \).

\( \Rightarrow X \) is first countable.

2) Let \( B \) be any open subcovering, for each \( n \), choose an element \( s \) \( \in \) \( A_n \) if possible.

Let \( B = \{A_n\} \) is a countable collection of \( B \).

Let, \( x \in X \),

\( x \in B \) for some \( B \in B \).

Since \( B \) is open set and \( \{B_n\} \) is basis,

\( \exists B_{n_0} \in \{B_n\} \) s.t \( x \in B_{n_0} \in B \)

\( \exists A_{n_0} \) s.t \( x \in B_{n_0} \subseteq A_{n_0} \subseteq B \).

\( \Rightarrow A \) is countable covering of \( X \).
\( \Rightarrow X \) is Lindelöf.

**iii.** Let \( x_n \in B_n \)

Let \( D = \{ x_1, x_2, \ldots \} \)

Then \( D \) is Countable. \( \Rightarrow \) \( \exists \) countable basis \( \mathcal{B} \).

To show that \( \overline{D} = X \), \( \mathcal{B} \) where \( X \) is second-countable.

Let \( x \in X \), let \( U \) be any \( \mathcal{B} \) of \( x \), \( \exists \) basis element \( B \) such that \( x \in B \cap U \).

\( \Rightarrow x_n \in U \) (\( x_n \in B_n \))

\( \Rightarrow U \cap D \neq \emptyset \)

\( \Rightarrow x \in \overline{D} \Rightarrow x \in \overline{D} \subseteq X \).

\( \Rightarrow \overline{D} = X \).

\( \Rightarrow D \) is dense in \( X \).

\( \Rightarrow x \) is separable.

Converse is not true.

Counterexample: \( \mathbb{Q} \) is not separable.

Consider \( \mathbb{R}^2 \).

**i.** Let \( x \in \mathbb{R}^2 \)

Let \( B_n = \{ x \mid x + \frac{1}{n} \in \mathbb{Z} \} \)

\( \Rightarrow B_n \) is Countable Collection of \( \mathbb{N} \) of \( x \)

Let \( U \) be any \( \mathcal{B} \) of \( x \).

\( \Rightarrow (x_1, a) \in U \) for some \( a \neq x \).

\( \exists \) \( n \) such that \( x_n < x = \frac{1}{y} \) for some \( y < x \).

\( \Rightarrow a = x + y \), \( x > 0 \)

\( \Rightarrow (x, x + x) \in (x, x+y) = \mathbb{R}^2 \)

\( \Rightarrow \mathbb{R}^2 \) is first-countable.

**ii.** \( \Rightarrow \mathbb{R}^2 \) is Lindelöf (from look P. N. 150).

**iii.** \( \mathbb{R}^2 \) is separable.
Let \( \mathbb{Q} \) be rationals, \( \mathbb{Q} \) is Countable.

Let \( x \in \mathbb{R}^2 \), let \( x \in (a, b) \)

Let \( \mathcal{B} \) be a basis for \( \mathbb{R} \).

Let \( x \in \mathbb{R}^2 \)

\[ \Rightarrow \exists (x, x+1) \text{ is open} \]

Let \( B_x \in \mathcal{B} \) s.t. \( x \in B_x \).

Let \( y \in \mathcal{B} \) s.t. \( x \in B_y \).

Since \( x+y \), \( B_x \neq B_y \).

There are uncountably many elements in \( \mathbb{R} \).

Hence \( B \) must have uncountably many members.

\( \mathbb{R}^2 \) does not have Countable basis.

\( \mathbb{R}^2 \) is not Second Countable.

**Note:**

1. Metrizable + Lindelof \( \Rightarrow \) Second-countable
2. Metrizable + Separable \( \Rightarrow \) Second-countable
3. Metrizable + Compact \( \Rightarrow \) Second-countable.

Ex. Product of Lindelof spaces need not be Lindelof.

**Solution:** \( \mathbb{R}^2 \) is Lindelof.

To show that \( \mathbb{R}^2 \) is not Lindelof, let \( L = \left\{ x(x, x) / x \in \mathbb{R} \right\} \).

L is closed in \( \mathbb{R}^2 \).
Let \( (a_i - a) \in L \)

Let \( (a_i, b_i) \times (c_i, d_i) \) be an open neighborhood of \( (a_i - a) \) in \( \mathbb{R}^2 \)
as shown in above fig:

This contains exactly \( \{(a_i, a) \} \) of \( L \).

Let \( A \) be a collection of all such \( (a_i, b_i) \) and \( (c_i, d_i) \).

\[ \{ (a_i, b_i) \} \quad \text{C: } L \text{ is closed} \Rightarrow 1 (a_i, b_i) \text{ is open} \]

\[ \cap \quad \text{is open covering of} \ 1 (a_i, b_i) \]

Since \( L \) is uncountable and each member of \( A \) contains at most one point of \( L \), no countable

subcollection of \( A \) can cover \( L \) and hence can

cover \( 1 (a_i, b_i) \) \[ \Rightarrow \]

\( 1 (a_i, b_i) \) is not Lindelof.

Example: Subspace of Lindelof need not be Lindelof.

Solution: Order square \( I^2 \) is compact.

\[ \Rightarrow I^2 \text{ is Lindelof. } \]

(where \( I = [0, 1] \))

Let \( A = I^2 \times (0, 1) \); be a subset of \( I^2 \)

\[ A = \bigcup \{(x, x, 0, 1) \times E \} \]

Note that \( \{(x, x, 0, 1) \} \) is open in \( A \)

\[ \Rightarrow A = \{ (x, 0, 1) \times E \} \text{ is open covering of } A \]
No countable subcollection of 8 can cover A. 

A is not Lindelöf. 

Subspace of Lindelöf space need not be Lindelöf. 

Example: Show that $\mathbb{R}_0$ and $\mathbb{R}_0^2$ are not metrizable. 

Solution: Suppose $\mathbb{R}_0$ is metrizable. 

We know that $\mathbb{R}_0$ is separable. 

$\mathbb{R}_0$ is second countable. 

($\because$ metrizable $\Rightarrow$ separable $\Rightarrow$ second countable) 

Which is contradiction to $\mathbb{R}_0$ is not second countable. 

If $\mathbb{R}_0^2$ is metrizable, 

we know $\mathbb{R}_0^2$ is compact. 

$\Rightarrow$ $\mathbb{R}_0^2$ is second countable. 

$\Rightarrow$ compact + metrizable $\Rightarrow$ second countable. 

$\mathbb{R}_0$ is not metrizable. 

If $\mathbb{R}_0^2$ is metrizable, we know $\mathbb{R}_0^2$ is compact. 

$\Rightarrow$ $\mathbb{R}_0^2$ is second countable. 

$\Rightarrow$ compact + metrizable $\Rightarrow$ second countable. 

Joint $\Rightarrow \mathbb{R}_0^2$ is Lindelöf. 

and $\mathbb{R}_0$ is Lindelöf. 

Every second countable space is Lindelöf. 

We know every subspace of second countable space is second countable. 

$\Rightarrow$ every subspace of Lindelöf space is Lindelöf. 

but $A = \{ (x, x \cos x) \mid x \in \mathbb{R} \}$ is a subspace of Lindelöf space. $\mathbb{R}_0^2$ is not Lindelöf. 

Which is not Lindelöf. 

$\Rightarrow$ a contradiction. 

$\Rightarrow$ $\mathbb{R}_0^2$ is not metrizable. 

Example: Let $X$ have a countable basis. Let $A$ be an uncountable subset of $X$. Show that uncountably many points of $A$ are limit points of $A$. 

$\Rightarrow$
Solution:

Let \( B = \{ B_n \} \) be countable basis for \( X \).

Suppose \( x \in A \) is not limit point of \( A \).

Choose \( B_n \in B \) such that \( B_n \cap A = \{ x \} \).

If \( x \neq y \in A \), \( y \) is not limit point of \( A \).

Let \( \theta \in \{ B_n \} \) be point of \( A \) which are not limit points of \( A \).

Remaining uncountable elements are limit points of \( A \).

Example:

Let \( A \) be a closed subspace of \( X \). Show that if \( X \) is Lindelöf, then \( A \) is Lindelöf.

Solution:

Let \( X \) is Lindelöf, \( A \) is closed subspace of \( X \).

Let \( \theta \) be open covering of \( A \).

\( \Rightarrow B = \theta \cup \{ x \in A \} \) is open covering of \( X \).

(Show that if \( A \) is closed \( \Rightarrow X \cap A \) is open in \( X \).)

\( \Rightarrow B \) has a countable subcollection \( B' \) which covers \( X \).

(\( X \) is Lindelöf)

\( \Rightarrow B' \leq B \) has countable covers of \( A \).

Example:

Show that if \( X \) is a countable product of separable spaces having countable dense subsets, then \( X \) has a countable dense subset.

Solution:

Let \( X_1, X_2, \ldots \) be separable spaces.

\( \Rightarrow \exists ) \) countable \( A_i \subseteq X_i \) such that \( A_i = X_i \) for each \( i \).

Each \( A_i \) is countable limit point \( A \) is countable.
Let \( x = \prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot x_3 \cdot \cdots \cdot x_n \).

\[ A = \prod A_i = \prod A_i = \prod x_i = x. \]

\[ \rightarrow A \text{ is countable, dense subset of } x, \]

\[ \rightarrow x \text{ is separable.} \]

**Example:** Let \( f: x \rightarrow y \) be continuous, show that if

- \( x \) is Lindelöf, or if \( x \) has a countable dense subset

then \( f(x) \) satisfies the same condition.

\( \therefore \) i) \( x \) is Lindelöf \( \Rightarrow \) \( f(x) \) is Lindelöf.

ii) \( x \) is separable \( \Rightarrow \) \( f(x) \) is separable.

**Solution:** Let \( f: x \rightarrow y \) be continuous.

1. Let \( A = \{ \forall x \} \) be an open covering of \( f(x) \).

\[ B = \left\{ f(C_v) \right\} \text{ is an open covering of } f(x) \]

but \( x \) is Lindelöf

\[ \Rightarrow \text{ a countable subset } B' \text{ of } B \text{ is a covering of } x. \]

Let \( B' = \left\{ f(C_{v_1}), f(C_{v_2}) \right\} = \left\{ f(C_{v_i}) \right\} \text{ s.t. } x \in B' \Rightarrow x \in f(A). \)

Let \( A' = \left\{ v_i \right\} i = 1, 2, \ldots \) is countable covering of \( f(x) \)

\[ \Rightarrow f(x) \text{ is Lindelöf.} \]

ii) Suppose \( x \) has a countable dense subset \( A \).

\[ \Rightarrow f(A) \text{ is countable because } A \text{ is countable.} \]

Given a nbd. \( U \) of \( y \in f(x) \) in \( Y \), the set \( f(U) \) is open in \( X \) and a nbd. of the points in \( f'(f(U)) \).

By hypothesis, \( f(U) \) intersects \( A \) at some point \( a \).

Since \( x \in f(C_U) \), it follows that \( f(C_a) \in U \), so \( y \) is in
The closure of \( f(A) \). Because \( f(A) = f(x) \), the set \( f(A) \) is countably dense.

\[ f(x) \text{ is separable.} \]

**The Separation Axioms:**

**Definition:** Suppose that one-point sets are closed in \( X \). The space \( X \) is regular if for each pair consisting of \( X \) and a closed set \( B \) with \( x \notin B \), \( \exists \) disjoint open sets \( U, V \) such that \( x \in U \), \( B \in V \).

The space \( X \) is normal if for each pair of disjoint closed sets \( A, B \) \( \exists \) disjoint open sets \( U, V \) such that \( A \subseteq U \) and \( B \subseteq V \).

**Example:** A Normal Space \( \Rightarrow \) Regular Space \( \Rightarrow \) Hausdorff Space \( \Rightarrow \) T₁-space

1) A Normal Space \( \Rightarrow \) Regular Space.

Suppose \( X \) is normal.

Let \( x \in X \), \( B \) is closed in \( X \) with \( x \notin B \).

Let \( A = \{x\} \). Then \( A \) is closed in \( X \).

Since \( X \) is normal, \( \exists \) disjoint open sets \( U, V \) such that \( A \subseteq U \), \( B \subseteq V \).

Converse is not true.

\( \mathbb{R}_1 = \mathbb{R}_2 \times \mathbb{R}_2 \) is regular but not normal.

2) Regular Space \( \Rightarrow \) Hausdorff Space.

Suppose \( X \) is a regular space.

Let \( x \neq y \) be any two distinct points in \( X \).
Consider \( B = \{ y \} \) and \( X \) be a regular space. Define \( B \subseteq X \) with \( x \notin B \) and \( B \) is closed in \( X \) with \( x \notin B \).

By definition of a regular space, \( X \) and \( B \) are disjoint open sets containing \( x \) and \( B \) respectively, i.e., \( x \in U \) and \( U \subset X \) and \( B \cap U = \emptyset \).

\[ X \text{ is Hausdorff space.} \]

Converse is not true. If \( X \) is Hausdorff space but not regular,

**Theorem:** Let \( X \) be a topological space. Let one-point sets in \( X \) be closed.

1. \( X \) is regular Iff given a point \( x \) of \( X \) and a neighborhood \( U \) of \( x \) there is a neighborhood \( W \) of \( x \) such that \( W \subseteq U \).
2. \( X \) is normal Iff given a closed set \( A \) and an open set \( U \) containing \( A \), there is an open set \( V \) containing \( A \) such that \( \overline{A} \cap U = \emptyset \).

**Proof:**

1. Suppose \( X \) is regular. Let \( x \in X \) and \( U \) be a nbhd of \( x \), and \( A = X - U \). Let \( A = X - U \) be a closed nbhd of \( x \). Then \( x \) is in \( A \).

Further, \( x \notin A \) (since \( x \in U \)). Let \( x \) be a point in \( X \).

Let \( x \in X \) and \( A \neq U \). Since \( V \cap W = \emptyset \), \( V \cap W = \emptyset \) for all \( x \) in \( X \).

Let \( y \in U \). \( y \) is a nbhd of \( y \) such that \( W \cap Y = \emptyset \). Therefore \( y \notin U \).

Conversely, suppose \( X \) is regular. Let \( x \) be a point in \( X \). To show that \( X \) is regular.

Let \( x \in X \) and \( B \) is closed in \( X \) with \( x \notin B \).
$B^c$ is open in $X$. Further, if $x \in B^c$, then $x \in V$. By hypothesis, I open set $V$ such that $x \in V$, $V \in B^c$. Since $V$ is open, we have:

$\Rightarrow (B^c)^c \subseteq (V)^c$.

$= B \subseteq \emptyset$, where $H = (V)^c$ is open.

We have $W = (V)^c \Rightarrow W \cap V = \emptyset$.

$\Rightarrow W \cap y = \emptyset$.

$\Rightarrow x$ is regular.

b) Proof of this is same as the proof of part a).

Theorem:

b) A subspace of a Hausdorff space is Hausdorff.

A product of a Hausdorff space is Hausdorff.

b) A subspace of a regular space is regular, a product of regular spaces is regular.

Proof:

a) A subspace of a Hausdorff space is Hausdorff. A product of Hausdorff spaces is Hausdorff.

$\Rightarrow$ proof is already given in Notes 1.

b) Let $X$ be a regular space and $Y$ be a subspace of $X$.

Let $x \in Y$ and $B$ be a closed subset of $Y$ such that $x \notin B$. Let $B^c$ be open.

$\Rightarrow$ closure of $B$ in $Y = \bar{B} \cap Y = B \cap Y$.

$B^c$ is closed in $X$.

If $x \notin \bar{B}$, then $x \notin \bar{B} \cap Y = B \cap Y$.

$\Rightarrow$ a contradiction.

Since $x \in Y$, $Y$ is regular, $B$ disjoint open sets $U, V$ such that $x \in U$, $B \subseteq V$.

They are disjoint open sets.

$x \in U \cap V$.

$Y$ is regular.

$x \in Y$.
Subspace of regular space is regular.

Let \( X = \prod X_x \) where each \( X_x \) is regular.

To show that \( X \) is regular.

Let \( \bar{x} \in X \) and \( U \) be a nbd. of \( \bar{x} \) in \( X \).

We have \( \bar{x} = (x_x)_{x \in J} \).

\[ \forall x \in J, \quad \bar{x} = x_x, \quad U_x \text{ open in } X_x \text{ and } U_x = X_x \text{ all but } \text{finitely many } x \text{'s}. \]

Also \( \bar{x} \in U_x \) in \( X_x \).

Since \( X_x \) is regular, \( \exists \ V_x \) open in \( X_x \) such that \( x_x \in V_x \) and \( V_x \subseteq U_x \).

When \( U_x = X_x \), choose \( V_x = X_x \).

\[ \Rightarrow V_x = X_x = \bar{x} \subseteq U. \]

\[ \Rightarrow x = \bar{x} \text{ is closed in } X. \]

Let \( V = \prod V_x \).

Then \( V \) open in \( X \).

\[ \forall \bar{x} \in V, \quad \bar{x} \in V \subseteq U. \]

\[ \Rightarrow \bar{x} \text{ is } \text{regular}. \]

Product of regular space is regular.

Example: Show that \( \mathbb{R}^\mathbb{Q} \) is normal.

Solution: Let \( \bar{x} \in \mathbb{R} \); \( \bar{x} \) is closed in \( \mathbb{R} \).

\[ \Rightarrow \mathbb{R} - x \bar{Q} \text{ is open in } \mathbb{R}. \]

\[ \Rightarrow \mathbb{R} - x \bar{Q} \text{ is open in } \mathbb{R}. \]

(\( \mathbb{Q} \) topology on \( \mathbb{R} \) is finer than standard topology on \( \mathbb{R} \)).

\[ \Rightarrow x \bar{Q} \text{ is closed in } \mathbb{R}. \]

Let \( A, B \) be two disjoint, closed sets in \( \mathbb{R} \).
For \( a \in A \), choose \( (a, e_a) \) such that it does not intersect \( B \).

For \( b \in B \), choose \( e_b \) such that \( (b, e_b) \) does not contain any element of \( A \).

Let \( U = \bigcup_{a \in A} e_a \) and \( V = \bigcup_{b \in B} e_b \).

\( U \cap V \) are open, \( A \subseteq U \), \( B \subseteq V \), and \( U \cap V = \emptyset \).

\( \mathbb{R}_2 \) is normal.

*Note:* Product of normal spaces need not be normal. e.g., \( \mathbb{R}_2 \times \mathbb{R}_2 \) is not normal. (\( \star \) \( \mathbb{R}_2 \) is normal)

**Normal Spaces**

**Theorem:** Every regular space with a countable basis is normal.

i.e., regular + second-countable \( \Rightarrow \) normal.

**Proof:** Suppose \( X \) is regular space and let \( B \) be a countable basis for \( X \).

Let \( A, B \) be disjoint closed sets in \( X \).

Want: disjoint, open \( U, V \) such that \( A \subseteq U \), \( B \subseteq V \).

Let \( x \in A \); a neighborhood \( U \) of \( x \) disjoint from \( B \).

\( (x, x) \) is regular.

By \( P \), \( (P \| 151 \), (a)) \), \( \exists \) a mid. \( y \) of \( x \) such that \( y \in U \).

\( \forall e \in B \), \( \exists a \) basis element \( C \subseteq B \), such that \( C \subseteq U \).

Let \( e \subseteq V \subseteq V \subseteq U \), \( U \cap B = \emptyset \).

\( \Rightarrow \quad U \cap B = \emptyset \).

Since \( B \) is countable, \( \{C_n\}_n \) the subcollection of such \( C \) is (basis element) countable.

We can denote such collection by \( \\{C_n\}_n \).

Note that \( U \cap B = \emptyset \) and \( A \subseteq U \).
Similarly, if a countable subcollection \( \{ V_n \} \) of \( B \) s.t. \( V_n \cap A = \emptyset \) and \( B \subseteq \bigcup V_n \),

\[
V_n = V_n - U_{i=1}^n C_i
\]

\( \bar{V}_i \) is closed \( \Rightarrow \) finite union \( \bigcup_{i=1}^n \bar{V}_i \) is closed

\[
\Rightarrow x - U_{i=1}^n \bar{V}_i \text{ is open}
\]

\[
\Rightarrow \bigcap_{i=1}^n (x - U_{i=1}^n \bar{V}_i) = x \cap B \text{ is open}
\]

Similarly, each \( V_n' \) is open.

Let \( U = \bigcup_{i=1}^n C_i \), \( V_n' = \bigcup_{n=1}^r V_n' \).

\( U, V \) are open.

Let \( x \in A \Rightarrow x \in C_n \) for some \( n \).

\[
\Rightarrow x \in \bigcup_{i=1}^n \bar{V}_i \neq C_n
\]

\[
\Rightarrow x \in U \Rightarrow A \subseteq U
\]

Similarly, \( B \subseteq V \).

Assume that \( z \in U \cap V \).

\( z \in C_j, z \in V_k \) for some \( j \) and \( k \).

Suppose that \( j < k \).

\( z \in C_j \cap V_k \Rightarrow z \notin \bigcup_{i=1}^{j-1} C_i \) and \( z \notin \bigcup_{i=1}^{k-1} V_i \).

\( z \notin \bar{C}_j \) and \( z \notin \bar{V}_k \).

\( \Rightarrow \) a contradiction.

If \( j = k \), we get a similar contradiction.

\( \Rightarrow U \cap V = \emptyset \)

\( \Rightarrow x \) is normal.
Theorem: Every metrizable space is normal.

Proof: Let $(X,d)$ be a metric space.
Let $A$, $B$ be disjoint closed sets in $X$.
Let $a \in A$, choose $E_a > 0$ such that
$B(a, E_a) \cap B = \emptyset$.
Let $b \in B$, choose $E_b > 0$ such that
$B(b, E_b) \cap A = \emptyset$.
Let $U = \bigcup_{a \in A} B(a, E_a/2)$, $V = \bigcup_{b \in B} B(b, E_b/2)$.

$\Rightarrow U, V$ are open.
$A \subseteq U$ and $B \subseteq V$.

Suppose $z \in U \cap V$.

$z \in B(a, E_a/2]$ and $z \in B(b, E_b/2)$.

and $d(z, a) < E_a/2$, $d(z, b) < E_b/2$.

Consider $d(c(a, b)) = d(c(a_2, d) + d(z) + d(z, b)) < E_a + E_b/2$.

If $E_a \leq E_b$ then $E_a + E_b/2 \leq E_b + E_b = E_b$.

$\Rightarrow a \in B(c(b, E_b)). U \cap A = \emptyset$.

Therefore, a contradiction.

Similarly, if $E_a > E_b$, then $V \cap B = \emptyset$.

$b \in B(c(a, E_a))$.

Which is a contradiction to the fact that
$B \cap A = \emptyset$.

$U \cap V = \emptyset$.

Therefore, $X$ is normal.

$\Rightarrow X$ is normal.
Theorem: Every compact Hausdorff space is normal.

Proof: Let $X$ be a compact Hausdorff space. Let $A$ and $B$ be two disjoint closed sets in $X$. $A$, $B$ are compact.

C: Closed subspace of a compact space is compact.

Let $a \in A$, by lemma (26.4 in book). $A$ disjoint open sets $V_a, V_a$ such that $a \in V_a, B \subset V_a$.

$\Rightarrow \{V_a\}_{a \in A}$ is an open cover of $A$.

Since $A$ is compact, $\{V_a, V_a\}_{a \in A}$ covers $A$.

Let $U = V_{a_1} \cup V_{a_2} U \cdots \cup V_{a_k} \subset A$.

$V = V_{a_1} \cap V_{a_2} \cap \cdots \cap V_{a_k} \subset B$.

$U, V$ are open.

Let $z \in U \cap V \Rightarrow z \in U_{a_i}$ for some $i$.

$z \notin V_{a_i}$ $\forall i$ ($\Rightarrow V_{a_i} \cap U_{a_i} = \emptyset$).

$\Rightarrow$ a contradiction.

$\Rightarrow U \cap V = \emptyset$.

Thus $U, V$ are open disjoint, $A \subset U$, $B \subset V$.

$\Rightarrow X$ is normal.

---

Theorem: Every well ordered set $X$ is normal in the order topology.

Proof: Let $x \in X$. Want any interval of the form $(y, x]$ is open in order topology of $X$.

If $x$ is largest element, then $(y, x]$ is always open.

Suppose $x$ is not largest element of $X$.

Since $X$ is well ordered, set and $(x, \infty)$ is non-empty.
\[
\begin{align*}
&\text{\(x\): \(x\) is not largest} \\
\Rightarrow &\quad y(x, y') \\
\Rightarrow &\quad (x, y') \text{ has a smallest element, say } y'. \\
\Rightarrow &\quad y' \text{ is next to } x. \\
\Rightarrow &\quad (y, x') = (y, y')^T \text{ open in order topology}. \\
&\text{Let } A, B \text{ be disjoint closed sets in } X. \\
&\text{Let } a_0 \text{ be smallest element of } X. \\
\text{Case } 1: &\quad a_0 \notin A \cup B. \\
&\text{Let } a \in A \Rightarrow a = a_0 \Rightarrow a_0 \text{ is not smallest element of } X. \\
&\exists \text{ element } x \in X \text{ such that } x \leq a_0. \\
\text{Choose } x_0 \in X \text{ such that } (x_0, a_0) \cap B = \emptyset. \\
\text{Similarly for each } b \in B, \text{ we can choose } x_b \in X \text{ s.t. } (x_b, b) \cap A = \emptyset. \\
\text{Let } U = \bigcup (x_0, a_0] \cup \bigcup (x_b, b] \quad \text{a.e.} \\
\quad \quad V = \bigcup (x_b, b]. \\
\Rightarrow &\quad U, V \text{ are open, } A \subseteq U, B \subseteq V. \\
\text{To prove that } U, V \text{ are disjoint.} \\
\text{Suppose not. Let } z \in U \cap V. \\
\Rightarrow &\quad z \in U \cup (x_0, a_0], \quad z \in U \cup (x_b, b]. \\
\Rightarrow &\quad z \in (x_0, a_0] \cup (x_b, b] \quad \text{for some } a \in A, b \in B. \\
\Rightarrow &\quad a' \neq b' \Rightarrow a' \neq b' \Rightarrow a' < b'. \\
\text{Assume that } a' < b'. \\
\Rightarrow &\quad x_a \in U \quad \text{and} \quad x_b \in V. \\
\Rightarrow &\quad a' \leq x' \quad \text{then} \\
\Rightarrow &\quad z \in (x'_a, a] \cup (x'_b, b] \quad \text{for some } x' \in X. \\
\Rightarrow &\quad a' \neq b' \quad \text{a contradiction.} \\
\Rightarrow &\quad a' \in (x'_b, b] \\
\Rightarrow &\quad a' \notin (x_b, b]. \\
\Rightarrow &\quad \text{a contradiction.} \\
\Rightarrow &\quad a' \neq b' \Rightarrow a' < b'. \\
\Rightarrow &\quad \text{we get similar contradiction.} \\
\Rightarrow &\quad a' > b'. \\
\Rightarrow &\quad \text{Thus } U \cap V = \emptyset.
\end{align*}
\]
Case ii. Suppose $a_0 \in A \cup B$.

Without loss of generality, assume that $a_0 \in A$.

Note that $(a_0, q_i)$, where $q_i$ is next to $a_0$.

$\Rightarrow (a_0, q_i)$ is open $\Rightarrow$ open.

$\Rightarrow X - (a_0, q_i)$ is closed.

Let $A_1 = A \cap (X - (a_0, q_i)) = A - (a_0, q_i)$. 

$\Rightarrow A_1$ is closed.

Also $a_0 \notin A \cup B$.

By Case i, $U \in \mathcal{V}$ open disjoint such that $A_1 \cap U \subseteq B$.

Let $U_1 = U \cup (a_0, q_i)$. 

$\Rightarrow U_1$ is open.

Also $A = A_1 \cup (a_0, q_i) \subseteq U \cup (a_0, q_i) = U_1$.

Further $a_0 \notin V = \emptyset$ if $a_0 \in V$, $\exists \ a_0 \in (x_0, b]$ for some $b \in B$.

$\Rightarrow \ U_1 \cap V = \emptyset$ $\Rightarrow$ a contradiction $\leftarrow$

$\Rightarrow X$ is normal.

Ex: What about $\mathbb{R}^J$, where $J$ is an uncountable set?

Ans: $\mathbb{R}^J$ is not normal.

Solution $\mathbb{R}^J$ compact and Hausdorff.

$\Rightarrow \mathbb{R}^J$ compact and Hausdorff.

$\Rightarrow \mathbb{R}^J$ is normal.

Let $A = (\mathbb{R}^J) \subseteq \mathbb{R}^J$.

Now $\mathbb{R}$ is homeomorphic to $(\mathbb{R}, J)$.

$\Rightarrow (\mathbb{R})^J$ is homeomorphic to $(\mathbb{R})^J$.

$\Rightarrow \mathbb{R}^J$ is homeomorphic to $(\mathbb{R}, J)$.

$\Rightarrow \mathbb{R}^J$ is normal.

$\Rightarrow (\mathbb{R}, J)$ is normal.

$\Rightarrow \mathbb{R}^J$ is not normal.

$\Rightarrow \mathbb{R}^J$ is not normal.

$\Rightarrow$ Subspace of normal space need not be normal.
Example. Closed subspace of normal space is normal.

Solution. Let \( X \) be a normal space and \( Y \) be a subspace.

Let \( A, B \) be disjoint closed sets in \( Y \), but \( Y \) is closed in \( X \). \( A \cap x \neq \emptyset \).

\[ \Rightarrow A, B \text{ are closed in } X \implies A \cap x \neq \emptyset \]

but \( X \) is normal.

\[ \Rightarrow \exists \text{ open disjoint } U, V \text{ such that } A \subseteq U, B \subseteq V \]

\[ \text{Let } U' = U \cap Y, \quad V' = V \cap Y \]

\[ \Rightarrow U', V' \text{ are open in } Y \]

\[ \quad (\because U, V \text{ are open in } X) \]

\[ U' \cup V' = U', V' = b \quad \Rightarrow \quad U' \cup V' = b \]

\[ A \subseteq U', \quad A \subseteq V' \implies A \subseteq U' \cup V' = b \]

and \( B \subseteq V', B \subseteq V' \implies B \subseteq U' \cup V' = b \)

\[ \Rightarrow Y \text{ is normal}. \]

The Urysohn Lemma:

Definition. \( A, B \subseteq X \implies \exists \text{ continuous } f : X \rightarrow [0, 1] \)

such that \( f(a) = 0, \quad f(b) = 1 \), and \( f(A) = [0, 1] \) and \( f(B) = [0, 1] \) that we say that \( A, B \) can be separated by continuous function.
Theorem: (Urysohn Lemma)

Let \( X \) be a normal space, let \( A \) and \( B \) be disjoint closed subsets of \( X \). Let \([a, b]\) be a closed interval in the real line. Then there exists a continuous map

\[ f: X \rightarrow [a, b] \]

such that \( f(x) = a \) for every \( x \) in \( A \), and \( f(x) = b \) for every \( x \) in \( B \).

Examples:
- Show that a connected normal space having more than one point is uncountable.

Solution:
- Suppose that \( X \) is a connected normal space with \( a \neq b \).

Let \( a, b \in X \) with \( a \neq b \).

- \( \{a\} \) and \( \{b\} \) are closed in \( X \) and disjoint.

By Urysohn Lemma, \( f \) is a continuous function.

\[ f: X \rightarrow [0, 1] \]

such that

\[ f(a) = 0, \quad f(b) = 1 \]

\[ f \] is a closed map on \( X \).

\( X \) is connected and \( f \) is continuous.

\( \Rightarrow \) \( f(X) \) is connected.

(Continuous images of connected spaces are connected)

\[ 0, 1 \in f(X) \]

\( \Rightarrow f^{-1}(0) \) must be connected.

so \( \{a\} \) must be an interval.

\[ \Rightarrow [0, 1] \subset f(X) \subset [0, 1] \]

\( \Rightarrow f(X) = [0, 1] \)

Since \( f(X) \) is uncountable.

\( \Rightarrow X \) must be uncountable.

Example:
- Show that a connected regular space having more than one point is uncountable.

Solution:
- \( X \) is a regular connected space with \( |X| \geq 2 \).

Assume that \( |X| \leq 1 \).

[Further discussion on the theorem and its implications could be added here.]
Let \( \mathcal{A} \) be an open covering of \( X \).

Let \( X = \{ x_1, x_2, \ldots, x_n \} \).

If \( x_i \in X \), then \( x_i \in U \) for some \( U \in \mathcal{A} \).

Select \( U_i \in \mathcal{A} \) such that \( x_i \in U_i \).

Then \( x_i \in U_i \) for all \( i \).

Thus, there is a countable subcollection of \( \mathcal{A} \) which covers \( X \).

Hence, \( X \) is Lindelöf.

A space \( X \) is normal if and only if it is first countable and regular.

A space \( X \) is uncountable if and only if it is Lindelöf.

This is a contradiction to \( X \) is countable.

Hence, \( X \) is uncountable.

---

**Definition:** A space \( X \) is completely regular if one point sets are closed in \( X \) and if, for each point \( x_0 \) and each closed set \( A \) not containing \( x_0 \), there is a continuous function \( f: X \to [0,1] \) such that \( f(x_0) = 1 \) and \( f(A) = 0 \).

E.g., \( \mathbb{R}^2 \) is completely regular.

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**Theorem:** (The Urysohn Metrization Theorem)
Every regular space \( X \) with a countable basis is metrizable.

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**Theorem:** (The Tietze Extension Theorem)
Let \( X \) be a normal space, and let \( A \) be a closed subspace of \( X \).

a) Any continuous map of \( A \) into the closed interval \([a, b] \) of \( \mathbb{R} \) may be extended to a continuous map of all of \( X \) into \([a, b] \).

---
b) Any continuous map of a into IR may be extended to a continuous map of all of X into IR.

- The Tykhonoff Theorem.

- Lemma (a): Let X be a set, and A be a collection of subsets of X having the finite intersection property. Then there is a collection D of subsets of X such that D contains A and D has the finite intersection property, and no other collection of subsets of X that properly contains D has this property.

- Lemma (b): Let X be a set, let D be a collection of subsets of X that is maximal w.r.t. the finite intersection property. Then:
  a) Any finite intersection of elements of D is an element of D.
  b) If A is a subset of X that intersects every element of D, then A is an element of D.

- Tykhonoff Theorem: An arbitrary product of compact spaces is compact in the product topology.

- Proof: Let X = \prod_{\alpha \in J} X_\alpha, where X_\alpha is compact for each \alpha \in J.

  Claim: If A is any collection of subsets of X with F.I.P. then \bigcap_{A \in A} A \neq \emptyset.

  By above lemma (a), if a collection D of subsets of X with F.I.P. such that, A \in D, and D is maximal w.r.t. F.I.P.
\[ \cap_\in \in A \cap_\in B \text{ for } \in \in \mathcal{D} \] 

\[ \Rightarrow \text{ It is sufficient to prove that } \cap_\in \in \mathcal{D} \neq \emptyset \]

For each \( \in \in J \), consider projection on

\[ T_\in : x \rightarrow x_\in \]

Consider \( \{ T_\in(CD) : \in \in \mathcal{D} \} \)

Let \( T_\in(\phi_1) \cap T_\in(\phi_2) \cap \ldots \cap T_\in(\phi_n) \) be any finite intersection; then as \( \mathcal{D} \) has F.I.P.,

\[ D_1 \cap D_2 \cap \ldots \cap D_n = \emptyset \]

Let \( \text{let } z \in D_1 \cap D_2 \cap \ldots \cap D_n \)

\[ \Rightarrow T_\in(z) \in T_\in(\{D_1 \cap D_2 \cap \ldots \cap D_n\}) \in T_\in(\{D_1 \cap D_2 \cap \ldots \cap D_n\}) \]

\[ \Rightarrow T_\in(\phi_1) \cap T_\in(\phi_2) \cap \ldots \cap T_\in(\phi_n) \neq \emptyset \]

\[ \Rightarrow \{ T_\in(CD) : \in \in \mathcal{D} \} \text{ also satisfies F.I.P.} \]

\( X_\in \) is compact

\[ \Rightarrow \cup T_\in(CD) = \emptyset \]

\[ \text{Let } z_\in \in \cap T_\in(CD) \]

\[ \Rightarrow z_\in \in X_\in \]

\[ \text{we prove that } \in \in \mathcal{D} \]

\[ \text{Let } x_\in \in \pi_\in^{-1}(U_\in), \text{ where } U_\in \text{ is open in } x_\in \]

\[ \Rightarrow \pi_\in(x_\in) = z_\in \in U_\in \]
FIP - Finite Intersection Property

$\Rightarrow u_b$ is nbd. of $x_b$.

but $x_b \in \bigcap_{D \in \mathcal{D}} \pi_b(D)$

$\Rightarrow x_b \in \pi_b(D)$, $A \in \mathcal{D}$.

$\Rightarrow u_b \cap \pi_b(D) \neq \emptyset$, $A \in \mathcal{D}$.

$\exists \gamma \in D$ s.t. $\pi_b(\gamma) = u_b \cap \pi_b(D)$

$\Rightarrow \gamma \in \pi^{-1}_b(u_b)$

$\Rightarrow \gamma \in \pi^{-1}_b(u_b) \cap D$.

$\Rightarrow \pi^{-1}_b(u_b) \cap D \neq \emptyset$, $A \in \mathcal{D}$

By above lemma (b).

$\pi^{-1}_b(u_b) \in \mathcal{D}$

Thus every subbasis element containing $x$ is in $\mathcal{D}$.

Let $W$ be a basis element containing $x$. $x$.

$\Rightarrow W$ is intersection of finitely many subbasis elements of $\mathcal{D}$.

$\Rightarrow W$ is intersection of finitely many members of $\mathcal{D}$.

By above lemma (b) (part (a)), $W \in \mathcal{D}$.

Consider $x$, let $D \in \mathcal{D}$.

If $W$ is any basis element with $x \in W$.

$\Rightarrow W \in \mathcal{D}$.

$\Rightarrow W \cap D \neq \emptyset$ (\(\because \mathcal{D} \) has FIP).

$\Rightarrow x \in \bigcap_{D \in \mathcal{D}} D$ true for each $D \in \mathcal{D}$.

$\Rightarrow x \in \bigcap_{D \in \mathcal{D}} D = \bigcap_{D \in \mathcal{D}} D \neq \emptyset$
To show that \( X \) is compact.

Let \( \mathcal{C} \) be a collection of closed subsets of \( X \) with F.I.P.

By claim \( \bigcap_{\mathcal{C}} \mathcal{C} = \emptyset \).

\[
\bigcap_{\mathcal{C}} \mathcal{C} = \bigcap_{\mathcal{C}} \left( \bigcap_{\mathcal{C}} \mathcal{C} \right) = \bigcap_{\mathcal{C}} \left( \bigcap_{\mathcal{C}} \mathcal{C} \right) = \bigcap_{\mathcal{C}} \mathcal{C} = \emptyset.
\]

\[\Rightarrow \quad X \text{ is compact.}\]

\[\text{\( \blacksquare \)}\]